PRINT Last name:________________ First name:________________

Signature:________________ Student ID:________________

Math 454 Final Exam, Fall 2005

I. Examples, Counterexamples and short answer. ($6\frac{2}{3}$ ea.) Do not give proofs, but clearly indicate your proposed example or counterexample, or short answer where appropriate.

1. Give an example of a simple graph G for which $\kappa'(G) \neq \delta(G)$ (edge connectivity is not equal to minimum degree).

2. Give an example of a closed walk W on a graph G, where W contains a cycle but cannot be partitioned into cycles.

3. Give an example of a tree T with an even number of vertices which does not have a perfect matching. For this tree, identify a vertex subset $S \subseteq V(T)$ which violates Tutte's condition: $\forall S \subseteq T, o(T - S) \leq |S|$.

4. Give an example of a connected graph for which $\alpha(G) \neq \beta'(G)$, and indicate the corresponding independent set and edge cover. (Hints: Recall that $\alpha(G)$ is the maximum size of an independent set of G, and $\beta'(G)$ is the minimum size of an edge cover of G, where an edge cover is a set $L \subseteq E(G)$ such that every vertex of G is incident to some edge of L.)

5. Find a *decomposition* of the graph below into at most 6 cliques.

6. Give an example of a graph with at least 8 vertices and chromatic number 4. Show a coloring the vertices with 4 colors.

II. Constructions and Algorithms. (10 pts ea.) Do not write proofs, but do give clear, concise answers.

7. In the graph G with vertices x, y as indicated below, the number of pairwise edge-disjoint x, y-paths is defined to be $\lambda'(x, y)$. Convert G to a (directed) network and find $\lambda'(x, y)$ by solving a flow problem. (Hint: what does an undirected edge become in the network?)

8. Determine if the following sequence is *graphic*. If so, give a simple graph with this degree sequence; if not, give a non-simple graph with this degree sequence.

$$5, 5, 5, 4, 4, 3, 3, 3, 2$$

9. **(a)** Starting with the graph H below, given with edge weights, construct the **directed** graph G as follows.
   - $V(G)$ is the set of spanning trees of H,
   - Given spanning trees $T, T' \in V(G)$, the directed edge TT' is in $E(G)$ if and only if
     (i) $(T - e) + f = T'$ for some pair of edges $e, f \in E(H)$, and
     (ii) $w(T) \geq w(T')$. (The weight of the tree does not increase.)

   (b) Give an interpretation and/or possible use for this graph.

III. Proofs. (10 pts ea.) Part of the score is determined by careful formatting of the proof (forward and reverse directions, assumptions, conclusions, stating whether you are proving by direct proof, contrapositive, contradiction, induction, etc.). Partial credit will be awarded for this as well.

Prove ***TWO*** of 10-12. Clearly indicate which proofs you want graded.

10. Prove by using an extremal argument that every regular simple graph G with degree k has a path of length k.

11. Prove by induction that every closed odd walk contains an odd cycle. Clearly state the basis, inductive hypothesis, and inductive step.

12. Prove that the construction in Problem 9 is guaranteed to yield a weakly connected graph G whenever H has at least one spanning tree. (Weakly connected means the underlying undirected graph is connected.)

Prove ***ONE*** out of 13-15. Clearly indicate which proof you want graded.

13. The *weight* of a vertex in the hypercube Q_k is the number of 1's in its label. Prove that for every perfect matching in Q_k, the number of edges matching words of weight i to words of weight $i+1$ is $\binom{k-1}{i}$ for $0 \leq i \leq k-1$. (Recall that two vertices $x_1x_2\cdots x_k, y_1y_2\cdots y_k \in \{0,1\}^k = V(Q_k)$ are adjacent iff they differ in exactly one position.)

14. Use Prüfer codes to count the number of trees on the vertex set $[n] = \{1,\ldots,n\}$ which have exactly $n-2$ leaves.

15. (Unfinished business.) Companies $c_1,\ldots,c_p$ send $m_1 \geq m_2 \geq \cdots \geq m_p$ representatives, respectively, to a conference. At the conference seminars $s_1,\ldots,s_q$ can accommodate up to $n_1 \leq n_2 \leq \cdots \leq n_q$ participants, respectively. The additional constraints are that:
    - No company can send two representatives to the same seminar,
    - all representatives must participate in a seminar, and
    - the seminars need not all be filled.

    Prove the direction we omitted in class: If for all $0 \leq k \leq p$ and $0 \leq l \leq q$, it holds that

$$\sum_{i=k+1}^{p} m_i + k(q-l) + \sum_{j=1}^{l} n_j \quad \geq \quad \sum_{i=1}^{p} m_i,$$

then there exists an assignment of representatives to groups that satisfies all of the constraints.