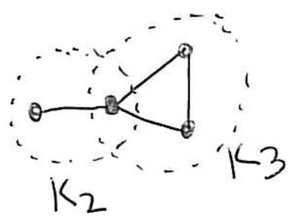
I. Examples and Counterexamples (5 points each). Do not give proofs, but clearly write a short answer or indicate your proposed example or counterexample.

1. Give examples of:
   (a) A tree with average degree 8/5.
   (b) A tree with average degree 5/3.

(a) P_5

(b) P_6

2. Draw an example of a graph with a maximal clique that is not the maximum clique of the graph. Clearly identify the two cliques.



K_2 maximal

K_3 maximum

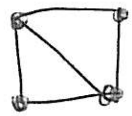
3. Give an example of a graph that cannot be expressed as the union of two bipartite subgraphs.

K_5

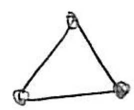
4. Give an example of an even closed walk that contains no cycle. Draw the graph and write out the standard vertex-edge list form of the walk.

W= u, e, v, e, u

5. Draw an example of a graph that contains a copy of C_4 as a subgraph, but does not contain a copy of C_4 as an induced subgraph.



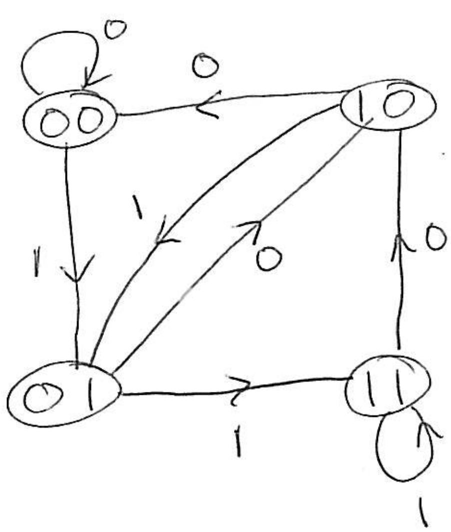
6. Give an example of a graph that is 3-partite but not 2-partite.



7. Give an example of a graph with smaller radius than diameter.

P_3

8. Draw the deBruijn graph that produces all binary strings of length 3 by recording edge labels on an Eulerian circuit (and looking at every possible consecutive 3 binary digits). Label all vertices and edges appropriately.

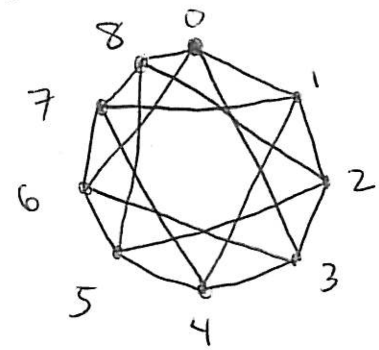


II. Constructions and Algorithms (15 points each). Do not write proofs, but do give clear, concise answers, showing the steps of any process or algorithm used.

9. Recall that for $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$, $a \bmod b = r$, where $0 \le r < b$ is obtained uniquely by the integer equation $a = bq + r$. Define the simple graph G as having vertex set and edge set

$$\begin{aligned} V(G) &= \{0,1,2,3,4,5,6,7,8\}, \qquad \text{and} \\ E(G) &= \{\{i,j\} : i,j \in V(G), (i-j) \bmod 9 \in \{1,3,6,8\}\}. \end{aligned}$$

(a) Draw G, labeling all vertices.

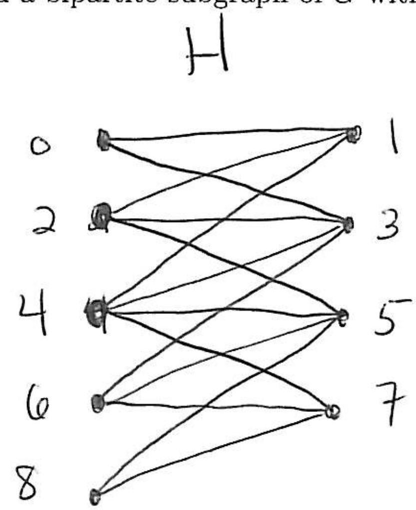


(c) Write down an Eulerian circuit in G by listing the vertices in order that they are visited by the circuit.

0, 3, 6, 0, 1, 4, 7, 1, 2, 5, 8, 2, 3, 4, 5, 6, 7, 8, 0

length 18

(d) Find a bipartite subgraph of G with at least $e(G)/2$ edges.

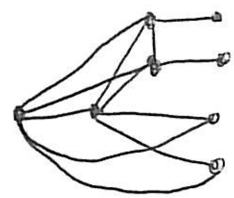


$$e(H) = 14 \geq \frac{18}{2} = \frac{e(G)}{2}$$

10. Determine whether the sequence $(5, 5, 4, 4, 2, 2, 1, 1)$ is the degree sequence of some simple graph. Show steps justifying why or why not, and draw a graph with this degree sequence if one exists.

$d_0 = 55442211$

$G_0 =$

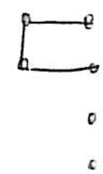


$d_1 = \text{sort}(4331111)$
$= 4331111$

$G_1 =$

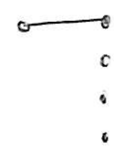
$d_2 = \text{sort}(220011)$
$= 221100$

$G_2 =$



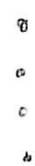
$d_3 = \text{sort}(10100)$
$= 11000$

$G_3 =$



$d_4 = 0000$

$G_4 =$



III. Proofs (10 points each). Partial credit for setting up a good proof structure without completing a proof.

11. Let $k \in \mathbb{Z}^+$, and assume G is a graph that has exactly $2k$ vertices with odd degree. Prove that there is no decomposition of G into $k-1$ or fewer paths.

Assume to contrary that $P_1, \ldots, P_j$ is a decomposition of G into $j \le k-1$ paths. WLOG, assume every P_i has length ≥ 1. Otherwise remove the trivial paths and reduce j. For all $v \in V(G)$, $d(v)$ is odd iff v appears as the endpoint of an odd number of these paths. Each path has exactly 2, distinct, endpoints. Therefore at most $2(j-1) < 2k$ vertices of G can be odd ※.

12. Let G be a loopless graph. Let H_1 and H_2 be maximal induced subgraphs of G such that both H_1 and H_2 have minimum degree ≥ 2. Prove that $H_1 = H_2$. (This proves that the maximal subgraph of G having minimum degree ≥ 2, called the 2-*core* of G, is unique.

Proof Assume to the contrary that $H \ne H_2$.

Then there exists $v \in V(H_1) \triangle V(H_2)$.

$G' = G[V(H_1) \cup V(H_2)]$ has minimum degree 2, since $d_{G'}(u) \ge d_{H_1}(u), d_{H_2}(u)$. for all u.

Therefore ~~neither~~ whichever H_i does not contain v is not maximal. ※

13. Let G be a connected graph, and let e be an edge of G. Prove that e is a cut-edge if and only if e belongs to every spanning tree of G. (If you use a result from the section without proof, be sure to quote it or describe it carefully.)

($\Rightarrow$) Let e be a cut-edge.
A spanning tree T of G is connected, with $V(T)=V(G)$,
and so $T \subseteq G-e$ is impossible, since G disconnecte
Therefore T contains e.

($\Leftarrow$) Assume e belongs to every spanning tree of G.
e is not a loop, since spanning trees contain no loops.
Suppose to the contrary that e is not a cut-edge.
Then e lies ~~not~~ on a~~ny~~ cycle C of G.
(From ~~2.1. e~~ a book: e a cut edge iff
e does not lie on any cycle.)
Since e is not a loop, C has length ≥ 2.
Let T be a spanning tree of G containing e.
T contains no cycle, and so
C has some edge f not on T.
By the exchange properties in 2.1, we may
choose f so that $T-e+f$ is a ~~tree~~.
spanning tree. But it does not contain e. ※