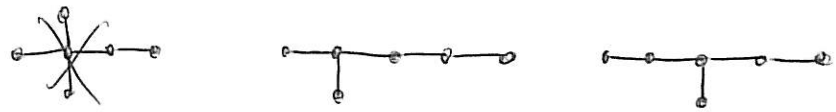
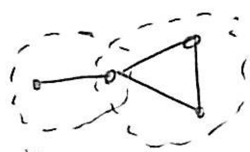
I. Examples and Counterexamples (5 points each). Do not give proofs, but clearly write a short answer or indicate your proposed example or counterexample.

1. Draw the non-isomorphic trees with average degree 5/3 and having exactly one vertex of degree 3.

   $n=6 \quad e=5.$



2. Draw an example of a graph with a maximal clique that is not the maximum clique of the graph. Clearly identify the two cliques.



   K_3 maximum

   K_2 maximal

3. Give an example of a graph that cannot be expressed as the union of three bipartite subgraphs.

   K_9

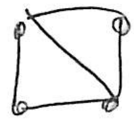
4. How many automorphisms are there on the graph C_5? (You may assume $V(C_5) = \{1, 2, 3, 4, 5\}$.)

   1) Choose $f(1)$ — 5 ways
   2) Choose wether $f(2) = f(1)+1$ or $f(2) = f(1)-1$ — 2 ways

   (arithmetic cycles from 5 to 1 + vice versa)

   10 automorphisms

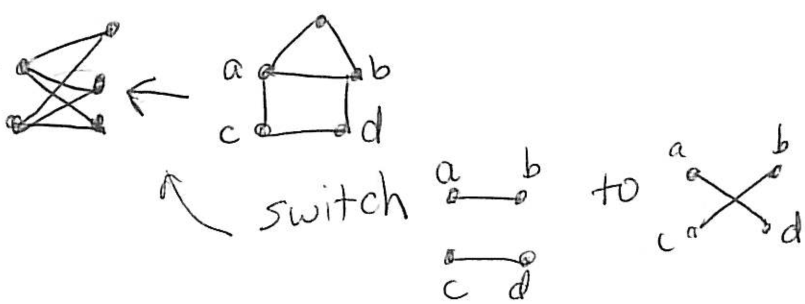
5. Draw an example of a graph that contains a copy of C_4 as a subgraph, but does not contain a copy of C_4 or K_4 as an induced subgraph.



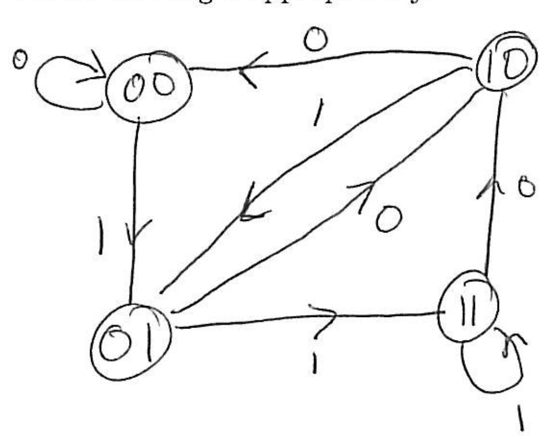
6. Let $k \in \mathbb{Z}^+$. Give an example of a graph that is $(k+1)$-partite but not k-partite.

$$K_{\underbrace{2,2,\dots,2}_{k+1 \text{ times}}} \qquad K_{\underbrace{1,1,\dots,1}_{k+1 \text{ times}}} = K_{k+1}$$

7. Convert the graph $K_{2,3}$ to the "house" graph (drawn on the whiteboard) using one or more 2-switches.



8. Draw the deBruijn graph that produces all binary strings of length 3 by recording edge labels on an Eulerian circuit (and looking at every possible consecutive 3 binary digits). Find an automorphism of this graph that is not the trivial identity automorphism. Label all vertices and edges appropriately.



nontrivial
automorphism:

$$f(b_1 b_2) = (1-b_1)(1-b_2)$$

(labels also flipped on edges)

II. Constructions and Algorithms (15 points each). Do not write proofs, but do give clear, concise answers, showing the steps of any process or algorithm used.

9. Recall that for $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$, $a \bmod b = r$, where $0 \leq r < b$ is obtained uniquely by the integer equation $a = bq + r$. Define the simple graph G as having vertex set and edge set

$$\begin{aligned} V(G) &= \{0,1,2,3,4,5,6,7,8\}, \qquad \text{and} \\ E(G) &= \{\{i,j\} : i,j \in V(G), (i-j) \bmod 9 \in \{1,3,6,8\}\}. \end{aligned}$$

(a) Draw G, labeling all vertices.

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(c) Write down an Eulerian circuit in G by listing the vertices in order that they are visited by the circuit.

(d) Find a bipartite subgraph of G with at least $e(G)/2$ edges.

10. Determine whether the sequence $(5, 5, 4, 4, 2, 2, 1, 1)$ is the degree sequence of some simple graph. Show steps justifying why or why not, and draw a graph with this degree sequence if one exists.

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III. Proofs (10 points each). Partial credit for setting up a good proof structure without completing a proof.

11. Let G be a graph, and let D be an orientation of G that is strongly connected. Prove that if G has an odd cycle, then D has an odd cycle. (Hint: what if some pair of consecutive vertices u, v on the cycle in G has only even u, v-paths in D?)

If G has a loop, we are done.

Otherwise let G have cycle $u_1 u_2 \cdots u_k$.

Property P_i: Suppose there exists a consecutive pair $u_i u_{i+1}$ (cycling at end) with an even u_i, u_{i+1}-path in D. If $u_{i+1} \to u_i$ in D, then we have an odd cycle with this edge completing the path.

If property P_i holds for <u>no</u> consecutive pair $u_i u_{i+1}$, then for <u>every</u> pair $u_i u_{i+1}$, either $u_i \to u_{i+1}$, or there exists an odd u_i, u_{i+1}-path.

Concatenate these odd paths all around the cycle (from G) of odd length to get an odd directed cycle in D. $\square$

12. Let G be a loopless graph. Let H_1 and H_2 be maximal induced subgraphs of G such that both H_1 and H_2 have minimum degree ≥ 2. Prove that $H_1 = H_2$. (This proves that the maximal subgraph of G having minimum degree ≥ 2, called the 2-*core* of G, is unique.

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13. Let G be a connected graph, and let e be an edge of G. Prove that e is a cut-edge if and only if e belongs to every spanning tree of G. (If you use a result from the section without proof, be sure to quote it or describe it carefully.)

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