Homework 10

Note: due to the adjusted schedule, recitation and homework due days will be as follows.
Recitation: 4/17, 4/21, 5/3
Homework: 4/19, 5/1, and 5/5

Recitation problems for Monday, 4/17/06

1. Show that the subadditivity inequality $\Pr(A_i \cup A_j) \leq \Pr(A_i) + \Pr(A_j)$ can give a substantial amount away compared to the independence assumption $\Pr(A_i \cap A_j) = \Pr(A_i)\Pr(A_j)$ by analyzing inequalities and/or giving a cogent example. An especially good response will find an example of events $A_1, \ldots, A_n$ such that
   - $\sum_i \Pr(A_i) \geq 1$,
   - $\Pr(\wedge_i \overline{A_i}) > 0$ under assumption of full mutual independence, and
   - $\Pr((\wedge_i \overline{A_i}) = c \in (0, 1)$ under assumption that each A_i is mutually independent from all but at most d of the other A_j's.

   The example can be given in terms of an explicit probability space, or by defining the dependency graph and other necessary quantities.

2. Prove the following conditional probability identity.
$$\Pr(\wedge_{i=1}^{n} \overline{A_i}) \;=\; \big(1 - \Pr(A_1)\big)\big(1 - \Pr(A_2|\overline{A_1})\big)\cdots\big(1 - \Pr(A_n|\wedge_{i=1}^{n-1}\overline{A_i})\big).$$

3. Prove the following conditional probability identity. Let $S = \{j_1, \ldots, j_r, l_1, \ldots, l_t\}$ equal the disjoint union $S = S_1 \dot\cup S_2$, where $S_1 = \{j_1, \ldots, j_r\}$ and $S_2 = \{l_1, \ldots, l_t\}$. Then
$$\begin{aligned}\Pr(\wedge_{j\in S_1}\overline{A_j} \mid \wedge_{l\in S_2}\overline{A_l}) \;=\; & (1 - \Pr(A_{j_1}\mid \wedge_{l\in S_2}\overline{A_l}))\cdot\big(1 - \Pr\big(A_{j_2}|\overline{A_{j_1}} \wedge (\wedge_{l\in S_2}\overline{A_l})\big)\big)\cdot\\ & \big(1 - \Pr\big(A_{j_3}|\overline{A_{j_2}} \wedge \overline{A_{j_1}} \wedge (\wedge_{l\in S_2}\overline{A_l})\big)\big)\cdot\\ & \vdots\cdot\\ & \big(1 - \Pr\big(A_{j_r}|\overline{A_{j_{r-1}}} \wedge \cdots \wedge \overline{A_{j_1}} \wedge (\wedge_{l\in S_2}\overline{A_l})\big)\big).\end{aligned}$$

4. Prove $(1 - 1/(d+1))^d > 1/e$ for all $d \geq 1$ (easy). Additionally, you should try to show the exact (real) range of d for which the inequality holds.

5. 4. Let $W(k)$ be the least integer n such that for any two-coloring of $\{1, 2, ..., n\}$, there is a monochromatic arithmetic progression of length k.
   (a) Use the basic probabilistic method to show that $W(k) \geq 2^{k/2}$.
   (b) Use the Lovasz Local Lemma to show that $W(k) > \dfrac{1}{2e}\dfrac{2^k}{k}(1 + o(1))$.

6. Problems 2, 3, 5, 6, p. 58-9 of Alon and Spencer.

7. Problem 4, p. 36 of Alon and Spencer.

8. Problems 3, 8, p. 21 of Alon and Spencer.

9. Problems 4, 7, p. 11 of Alon and Spencer.

10. Show that for any n sufficiently large, there exists a graph G on n vertices with chromatic number at least $n/2$ with clique number at most $n^{3/4}$. Chromatic number is the smallest number k such that the vertices of the graph can be partitioned into k parts with no edges inside any part. Clique number is the size of the largest clique (complete subgraph) in the graph. (Hint: What can you say about the chromatic number of the complement of a triangle-free graph?)

Written problems for Wednesday, 4/19/06

Work 1 of the following 3 problems using the Local Lemma. These should be done with only your notes and Alon and Spencer as a reference, plus the wikipedia entry for CNF.

1. #4, p.77 of Alon and Spencer.

2. (See `http://en.wikipedia.org/wiki/Conjunctive_normal_form` for a description of conjunctive normal form.) Prove that every conjunctive normal form Boolean expression which has exactly 10 distinct variables per clause and in which each variable appears in exactly 30 clauses has a satisfying assignment. You might consider the CNF form to be something like

$$\bigwedge_{i=1}^{n}\left(\bigvee_{j=1}^{10} v_{i,j}\right).$$

3. Let G be a simple (finite) graph on vertices V, and let $S(v)$ be a list of at least l colors for vertex $v \in V$. Suppose for each color $c \in S(v)$ there are at most $l/8$ neighbors u of v such that $c \in S(u)$. Prove that there is a coloring of V in which each v is colored from $S(v)$ and no two adjacent vertices $u, v \in V$ are colored with the same color.