Hypothesis Testing for Stochastic PDEs Driven by Additive Noise

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Joint work with Professor Cialenco

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- The Problem: Hypothesis Testing for the Drift of SPDEs

- Background and Introduction
  - Parameter Estimation
  - Continuous Observations in Time and Space
  - Maximum Likelihood Estimator

- Stochastic Heat Equations
  - Fractional Laplacian and Additive Noise
  - Maximum Likelihood Ratio
  - Strong Consistency and Asymptotic Normality
Simple Hypothesis for the Drift

- A Simple Result for Finite Time Observations
- Define Asymptotic Class of Tests
- Asymptotic Methods in Two Regimes (Increase Time and Fine Grid in Space)
- Technical Results on Large Deviation Principle
Consider the following stochastic heat equation:

\[ dU(t, x) - \theta U_{xx}(t, x)dt = dW(t, x), \quad x \in [0, \pi], t \in [0, T] \]

with zero initial condition: \( U(0, x) = 0 \) for all \( x \in [0, \pi] \), and zero boundary condition: \( U(t, 0) = U(t, \pi) = 0 \) for all \( t \in [0, T] \).

- given stochastic basis \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \)
- \( dW(t, x) = \sum_{k=1}^{\infty} h_k dw_k(t), \quad w_k(t), k \in \mathbb{N} \) independent Brownian Motions
- \( h_k = \sqrt{\frac{2}{\pi}} \sin(kx), k \geq 1 \) constitute a complete orthonormal base for \( L^2(0, \pi) \)
- \( U \) observed for all \( t \in [0, T] \) and all \( x \in [0, \pi] \) - continuous observations
\[ dU(t, x) - \theta U_{xx}(t, x)dt = dW(t, x), \quad x \in [0, \pi], \]

Based on the above assumptions, we consider the simple hypothesis:

\[ H_0 : \quad \theta = \theta_0, \]
\[ H_1 : \quad \theta = \theta_1. \]

For simplicity, assume \( \theta_1 > \theta_0 \).
\[ dU(t, x) - \theta U_{xx}(t, x)dt = \sum_{k=1}^{\infty} h_k dw_k(t), \quad U(0, x) = 0, \quad x \in [0, \pi], \]

- Taking the \( L^2 \) inner product of the equation with \( h_k(x) \), we obtain

\[ du_k(t) = -\theta k^2 u_k(t)dt + dw_k(t), \quad u_k(0) = 0, \quad t \geq 0, \quad k \geq 1, \]

where \( u_k(t) = \langle U(t, x), h_k(x) \rangle \).

- Solution:

\[ u_k(t) = \int_0^t e^{-\theta k^2(t-s)} dw_k, \quad k \geq 1. \]
Let $P_{\theta}^{N,T}(\cdot) = P(U_T^N \in \cdot)$ be the measure on $C([0, T]; \mathbb{R}^N)$ generated by $U_T^N(t) := (u_1(t), \ldots, u_N(t))$ up to time $T$.

$P_{\theta}^{N,T}$ are equivalent for different values of the parameter $\theta$, $P_{\theta}^{N,T} \sim P_{\theta_0}^{N,T}$ for $\theta \neq \theta_0$.

Apply Girsanov Theorem and get the ‘Likelihood Ratio’ (Radon-Nikodym derivative) $dP_{\theta}^{N,T}/dP_{\theta_0}^{N,T}$.

Maximize the Log-Likelihood Ratio

$$\hat{\theta}_T^N(\omega) := \arg \max_{\theta} \ln \frac{dP_{\theta}^{N,T}}{dP_{\theta_0}^{N,T}}(\omega),$$

which is called Maximum Likelihood Estimator.
Parameter Estimation

Maximum Likelihood Estimator

- Let $P^{N,T}_\theta (\cdot) = P(U^N_T \in \cdot)$ be the measure on $C([0, T]; \mathbb{R}^N)$ generated by $U^N_T(t) := (u_1(t), \ldots, u_N(t))$ up to time $T$.

- $P^{N,T}_\theta$ are equivalent for different values of the parameter $\theta$, $P^{N,T}_\theta \sim P^{N,T}_{\theta_0}$ for $\theta \neq \theta_0$

- Apply Girsanov Theorem and get the ‘Likelihood Ratio’ (Radon-Nikodym derivative) $dP^{N,T}_\theta / dP^{N,T}_{\theta_0}$

- Maximize the Log-Likelihood Ratio

$$\hat{\theta}^N_T(\omega) := \arg \max_{\theta} \ln dP^{N,T}_\theta / dP^{N,T}_{\theta_0}(\omega),$$

which is called Maximum Likelihood Estimator (MLE).
Likelihood Ratio is

\[ L(\theta_0, \theta, U_T^N) = \exp \left( - (\theta - \theta_0) \sum_{k=1}^{N} k^2 \right) \]

\[ \times \left( \int_{0}^{T} u_k(t) du_k(t) + \frac{1}{2} (\theta + \theta_0) k^2 \int_{0}^{T} u_k^2(t) dt \right). \]
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Maximizing with respect to \( \theta \) and get

\[ \hat{\theta}_T^N = - \frac{\sum_{k=1}^{N} \int_{0}^{T} k^2 u_k(t) du_k(t)}{\sum_{k=1}^{N} \int_{0}^{T} k^4 u_k^2(t) dt} \]
Theorem (X., Igor Cialenco)

1 (Consistency in large time asymptotics)
For every fixed $N \in \mathbb{N}$,

$$\lim_{T \to \infty} \hat{\theta}_T^N = \theta, \quad a.s. \quad (2.1)$$
Theorem (X., Igor Cialenco)

1. (Consistency in large time asymptotics)
   For every fixed $N \in \mathbb{N}$,
   \[
   \lim_{T \to \infty} \hat{\theta}^N_T = \theta, \quad \text{a.s.} \tag{2.1}
   \]

2. (Asymptotic normality in large time asymptotics)
   For every fixed $N \in \mathbb{N}$,
   \[
   \lim_{T \to \infty} \sqrt{T} \left( \hat{\theta}^N_T - \theta \right) \xrightarrow{d} \mathcal{N}(0, 2\theta/M), \tag{2.2}
   \]

   where $M = \sum_{k=1}^{N} k^2$. 
Theorem (Huebner and Rozovskii [1995], Lototsky [2009])

1. (Consistency in number of Fourier coefficients)
   For every fixed $T > 0$,
   \[
   \lim_{N \to \infty} \hat{\theta}_T^N = \theta, \quad \text{a.e.} \tag{2.3}
   \]

2. (Asymptotic normality in number of Fourier coefficients)
   For every fixed $T > 0$,
   \[
   \lim_{N \to \infty} N^{3/2} \left( \hat{\theta}_T^N - \theta \right) \xrightarrow{d} \mathcal{N} \left( 0, \frac{6\theta}{T} \right). \tag{2.4}
   \]
Now consider

\[
dU(t, x) + \theta(-\Delta)^\beta U(t, x)dt = \sigma \sum_{k=1}^{\infty} \lambda_k^{-\gamma} h_k(x) dw_k(t), \tag{2.5}
\]

where \( x \in G \), \( G \) is a bounded domain in \( \mathbb{R}^d \).

- \( t \in [0, T] \); zero initial and boundary values
- \( \Delta \) is the Laplace operator on \( G \) with zero boundary condition.
- \( \{ h_k \} \) are eigenfunctions in \( L^2(G) \); \( \{ \rho_k \} \) are eigenvalues; \( \lambda_k = \sqrt{-\rho_k} \)
- \( \theta > 0 \) (Unknown), \( \beta > 0 \), \( \gamma \geq 0 \), \( \sigma \in \mathbb{R} \setminus \{0\} \)
- consider solution in \( (H^\beta(G), H^0(G), H^{-\beta}(G)) \)
- \( U \) observed for all \( t \in [0, T] \) and all \( x \in G \) - continuous observations
Theorem (Existence and Uniqueness)

If $2(\gamma - s)/d > 1$, then (2.5) has a unique solution (weak in PDE sense, and strong in probability sense)
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U \in L_2(\Omega \times [0, T]; H^{s+\beta}(G)) \cap L^2(\Omega; C((0, T); H^s(G))).
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Theorem (Existence and Uniqueness)

If $2(\gamma - s)/d > 1$, then (2.5) has a unique solution (weak in PDE sense, and strong in probability sense)

$$U \in L_2(\Omega \times [0, T]; H^{s+\beta}(G)) \cap L^2(\Omega; C'((0, T); H^s(G))).$$

For more general setup and proof, see Rozovskii [1990], Chow [2007], Huebner et al. [1993, 1997], Lototsky [2009], Cialenco and Lototsky [2009].
- Likelihood Ratio:

\[ L(\theta_0, \theta, U_T^N) = \exp \left( - (\theta - \theta_0) \sigma^{-2} \sum_{k=1}^{N} \lambda_k^{2\beta+2\gamma} \right. \]

\[ \left. \times \left( \int_0^T u_k(t) du_k(t) + \frac{1}{2} (\theta + \theta_0) \lambda_k^{2\beta} \int_0^T u_k^2(t) dt \right) \right) \]

- Maximum Likelihood Estimator:

\[ \hat{\theta}_T^N = - \frac{\sum_{k=1}^{N} \lambda_k^{2\beta+2\gamma} \int_0^T u_k(t) du_k(t)}{\sum_{k=1}^{N} \lambda_k^{4\beta+2\gamma} \int_0^T u_k^2(t) dt}, \quad N \in \mathbb{N}, \quad T > 0. \]
Theorem (X., Igor Cialenco)

1. For every fixed $N \in \mathbb{N}$,
   \[
   \lim_{T \to \infty} \hat{\theta}_T^N = \theta, \quad \text{a.e.}
   \]  
   \hspace{0.5cm} (2.6)

2. For every fixed $N \in \mathbb{N}$,
   \[
   \lim_{T \to \infty} \sqrt{T} \left( \hat{\theta}_T^N - \theta \right) \xrightarrow{d} \mathcal{N}(0, 2\theta/M),
   \]  
   \hspace{0.5cm} (2.7)

where $M = \sum_{k=1}^{N} \lambda_k^2$. 
Theorem (Huebner and Rozovskii [1995], Lototsky [2009])

1. For every fixed $T > 0$,

$$
\lim_{N \to \infty} \hat{\theta}_T^N = \theta, \quad \text{a.e.}
$$

(2.8)

2. For every fixed $T > 0$,

$$
\lim_{N \to \infty} \frac{N^{\beta/d + \frac{1}{2}}}{d} \left( \hat{\theta}_T^N - \theta \right) \xrightarrow{d} \mathcal{N} \left( 0, \frac{(4\beta/d + 2)\theta}{\varpi^\beta \sigma^2 T} \right).
$$

(2.9)

Here $\varpi$ is defined by

$$
\varpi := \lim_{k \to \infty} |\rho_k| k^{-2/d}.
$$
Consider a simple hypothesis:

\[ H_0 : \theta = \theta_0, \]
\[ H_1 : \theta = \theta_1. \]

For simplicity, assume \( \theta_1 > \theta_0 \) and \( \sigma > 0 \).
Basic Setup

\[ dU(t, x) + \theta(-\Delta)^\beta U(t, x)dt = \sigma \sum_{k=1}^{\infty} \lambda_k^{-\gamma} h_k(x)dw_k(t), \quad U(0, x) = 0. \]

- Taking the $L^2$ inner product of the equation with $h_k(x)$, we obtain

\[ du_k = -\theta \lambda_k^{2\beta} u_k dt + \sigma \lambda_k^{-\gamma} dw_k(t), \quad u_k(0) = 0, \quad 1 \leq k \leq N. \]

where $u_k(t) = \langle U(t, x), h_k(x) \rangle$. 
Hypothesis Testing

Basic Setup

\[ \text{Basic Setup} \]

\[ dU(t, x) + \theta(-\Delta)^\beta U(t, x)dt = \sigma \sum_{k=1}^{\infty} \lambda_k^{-\gamma} h_k(x)dw_k(t), \quad U(0, x) = 0. \]

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where \( u_k(t) = \langle U(t, x), h_k(x) \rangle \).

- Let \( \mathbb{P}_{\theta, T}^N(\cdot) = \mathbb{P}(U_T^N \in \cdot) \) be the measure on \( C([0, T]; \mathbb{R}^N) \) generated by \( U_T^N(t) = (u_1, \ldots, u_N) \) up to time \( T \).
Construction of the Test

- Fix time $T$; Fix Fourier modes $N$.
- Look for rejection region $R \in \mathcal{B}(C([0, T]; \mathbb{R}^N))$.
- $\mathbb{P}^{N,T}_{\theta_0}(R)$, which we call Type I error, is the probability that we reject the null hypothesis when it is true.
- Define class

$$K_\alpha := \left\{ R \in \mathcal{B}(C([0, T]; \mathbb{R}^N)) : \mathbb{P}^{N,T}_{\theta_0}(R) \leq \alpha \right\}.$$ 

Here $\alpha \in (0, 1)$ is called the significant level. For example, we may take $\alpha = 0.1, 0.05$ or $0.01$.
- $\mathbb{P}^{N,T}_{\theta_1}(R)$ is called the power of the test, and $1 - \mathbb{P}^{N,T}_{\theta_1}(R)$ is called Type II error.
We say that a rejection region $R^* \in \mathcal{K}_\alpha$ is the most powerful in the class $\mathcal{K}_\alpha$ if

$$P_{\theta_1}^{N,T}(R) \leq P_{\theta_1}^{N,T}(R^*), \quad \text{for all } R \in \mathcal{K}_\alpha.$$
Most Powerful Test

**Definition**
We say that a rejection region $R^* \in \mathcal{K}_\alpha$ is the most powerful in the class $\mathcal{K}_\alpha$ if

$$P_{\theta_1}^{N,T}(R) \leq P_{\theta_1}^{N,T}(R^*), \quad \text{for all } R \in \mathcal{K}_\alpha.$$

**Natural Guess:**
Likelihood Ratio type tests should be the most powerful.
Theorem (X., Igor Cialenco)

(Neyman-Pearson lemma)

Take

\[ L(\theta_0, \theta_1, U^N_T) = \exp \left( - (\theta_1 - \theta_0) \sigma^{-2} \sum_{k=1}^{N} \lambda_k^{2\beta + 2\gamma} \right. \]
\[ \left. \times \left( \int_0^T u_k(t) du_k(t) + \frac{1}{2} (\theta_1 + \theta_0) \lambda_k^{2\beta} \int_0^T u_k^2(t) dt \right) \right). \]

Let \( c_\alpha \) be a real number such that

\[ \mathbb{P}^{N,T}_{\theta_0}(L(\theta_0, \theta_1, U^N_T) \geq c_\alpha) = \alpha. \] (3.1)

Then,

\[ R^* := \{ U^N_T : L(\theta_0, \theta_1, U^N_T) \geq c_\alpha \}, \] (3.2)

is the most powerful rejection region in the class \( K_{\alpha} \).
Did we solve the problem?

This result gives a complete theoretical answer to the hypothesis testing problem.
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Not really!

However, this result cannot be used in practice, since it is not possible to find $c_\alpha$ explicitly. Recall the relationship between $\alpha$ and $c_\alpha$:
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$$\mathbb{P}_{\theta_0}^{T,N}(L(\theta_0, \theta_1, U^N_T) \geq c_\alpha) = \alpha.$$

For finite $T$ and $N$, $c_\alpha$ has no explicit formula!
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For finite $T$ and $N$, $c_\alpha$ has no explicit formula!

We would suggest “Asymptotic Method”:

- Fix $N$, let $T \to \infty$
- Fix $T$, let $N \to \infty$
- Let both $T$ and $N$ go to infinity
Define a new class (Also see Kutoyants [2004])

\[ \mathcal{K}_\alpha := \left\{ (R_T)_{T \in \mathbb{R}_+} : R_T \in \mathcal{B}(C([0, T]; \mathbb{R}^N), \limsup_{T \to \infty} \mathbb{P}_{\theta_0}^{N,T}(R_T) \leq \alpha \right\}, \]

where \( N \) is fixed. \( \alpha \) is called “Asymptotic Significant Level”.
Define a new class (Also see Kutoyants [2004])

\[ K_\alpha^* := \left\{ (R_T)_{T \in \mathbb{R}_+} : R_T \in \mathcal{B}(C([0, T]; \mathbb{R}^N), \limsup_{T \to \infty} \mathbb{P}_{\theta_0}^{N,T}(R_T) \leq \alpha \right\} , \]

where \( N \) is fixed. \( \alpha \) is called “Asymptotic Significant Level”.

**Natural Extrapolation:**

Rejection region \((R_T^*)_{T \in \mathbb{R}_+}\) would be a good one if

\[ \lim_{T \to \infty} \mathbb{P}_{\theta_0}^{N,T}(R_T^*) = \alpha. \]
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\]

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Rejection region \((R_T^*)_{T \in \mathbb{R}_+}\) would be a good one if

\[
\lim_{T \to \infty} \mathbb{P}_{\theta_0}^{N,T}(R_T^*) = \alpha.
\]

Natural Question:

Does the Likelihood Ratio test still work? What is \( c_\alpha \)?
To find $c_{\alpha}$, we make the following heuristic argument: by Itô’s Formula,

\[
\mathbb{P}_{\theta_0}^{N,T}(L(\theta_0, \theta_1, U^N_T) \geq c^*_\alpha) = \mathbb{P}_{\theta_0}^{N,T} \left( X_T - \frac{2(\theta_1 + \theta_0)}{(\theta_1 - \theta_0)\sigma \sqrt{T}} Y_T \geq \frac{4\theta_0 \ln c^*_\alpha}{(\theta_1 - \theta_0)^2 T} + M \right),
\]

where

\[
M := \sum_{k=1}^{N} \lambda_k^{2/\beta}, \quad X_T := \sum_{k=1}^{N} \frac{\lambda_k^{2/\beta+2\gamma} u_k^2(T)}{\sigma^2 T},
\]

\[
Y_T := \frac{1}{\sqrt{T}} \sum_{k=1}^{N} \lambda_k^{2/\beta+\gamma} \int_0^T u_k dw_k.
\]
We can prove:

- And we have the split:

\[
\mathbb{P}^{N,T}_{\theta_0}(L(\theta_0, \theta_1, U^N_T) \geq c^*_\alpha) \leq \mathbb{P}^{N,T}_{\theta_0}(X_T \geq \delta) \\
+ \mathbb{P}^{N,T}_{\theta_0} \left( - \frac{2(\theta_1 + \theta_0)}{(\theta_1 - \theta_0)\sigma \sqrt{T}} Y_T \geq \frac{4\theta_0 \ln c^*_\alpha}{(\theta_1 - \theta_0)^2 T} + M - \delta \right).
\]

- For any fixed \( \delta > 0 \), \( \mathbb{P}^{N,T}_{\theta_0}(X_T \geq \delta) \to 0 \) as \( T \to \infty \).

- \( Y_T \overset{d}{\to} \mathcal{N}(0, \sigma^2 M/(2\theta_0)) \) as \( T \to \infty \).
We can prove:

- And we have the split:

\[
\mathbb{P}^N,T_{\theta_0} \left( L(\theta_0, \theta_1, U^N_T) \geq c^*_\alpha \right) \leq \mathbb{P}^N,T_{\theta_0} (X_T \geq \delta) \\
+ \mathbb{P}^N,T_{\theta_0} \left( - \frac{2(\theta_1 + \theta_0)}{(\theta_1 - \theta_0)\sigma \sqrt{T}} Y_T \geq \frac{4\theta_0 \ln c^*_\alpha}{(\theta_1 - \theta_0)^2 T} + M - \delta \right).
\]

- For any fixed \( \delta > 0 \), \( \mathbb{P}^N,T_{\theta_0} (X_T \geq \delta) \to 0 \) as \( T \to \infty \).

- \( Y_T \xrightarrow{d} \mathcal{N}(0, \sigma^2 M/(2\theta_0)) \) as \( T \to \infty \).

It Is Reasonable To Take:

\[
- \sqrt{\frac{2\theta_0}{M}} \frac{(\theta_1 - \theta_0)\sqrt{T}}{2(\theta_1 + \theta_0)} \left[ \frac{4\theta_0 \ln c^*_\alpha}{(\theta_1 - \theta_0)^2 T} + M \right] = q_\alpha. \tag{3.3}
\]
Solve (3.3) to get

\[ c^\#_\alpha(T) = \exp \left( -\frac{(\theta_1 - \theta_0)^2}{4\theta_0} MT - \frac{\theta_1^2 - \theta_0^2}{2\theta_0} \sqrt{\frac{MT}{2\theta_0}} q_\alpha \right). \] (3.4)
Solve (3.3) to get

\[ c^\#_\alpha(T) = \exp \left( -\frac{(\theta_1 - \theta_0)^2}{4\theta_0} MT - \frac{\theta_1^2 - \theta_0^2}{2\theta_0} \sqrt{\frac{MT}{2\theta_0}} q_\alpha \right). \]  

(3.4)

**Theorem (X., Igor Cialenco)**

**Suppose**

\[
R^\#_T := \{ U^N_T : L(\theta_0, \theta_1, U^N_T) \geq c^\#_\alpha(T) \}, \quad \text{for all } T,
\]

where \( c^\#_\alpha \) is given by (3.4). Then, the rejection region \( (R^\#_T)_{T \in \mathbb{R}^+} \in \mathcal{K}_{\alpha}^* \), and moreover

\[
\lim_{T \to \infty} \mathbb{P}_{\theta_0}^{N,T} (R^\#_T) = \alpha.
\]
The Next Question:

How does the power of this test $P_{\theta_1}^{N,T}(R_T^\#)$ behave?
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Recall that

For every fixed \( N \in \mathbb{N} \),

\[
\hat{\theta}_T^N \to \theta, \quad \text{a.e. as } T \to \infty.
\]
The Next Question:

How does the power of this test \( P_{\theta_1}^{N,T}(R_T^\#) \) behave?

Recall that

For every fixed \( N \in \mathbb{N} \),

\[ \hat{\theta}_T^N \to \theta, \quad \text{a.e. as } T \to \infty. \]

We should expect

\[ P_{\theta_1}^{N,T}(R_T^\#) \to 1, \quad \text{as } T \to \infty. \]
The Next Question:

How does the power of this test $\mathbb{P}_{\theta_1}^{N,T}(R_T^#)$ behave?

Recall that

For every fixed $N \in \mathbb{N}$,

$$\hat{\theta}_T^N \rightarrow \theta, \quad \text{a.e. as } T \rightarrow \infty.$$ 

We should expect

$$\mathbb{P}_{\theta_1}^{N,T}(R_T^#) \rightarrow 1, \quad \text{as } T \rightarrow \infty.$$ 

Theorem (X., Igor Cialenco)

$$1 - \mathbb{P}_{\theta_1}^{N,T}(R_T^#) \sim \exp(-I(\theta_0, \theta_1, N)T + o(T)), \quad \text{as } T \rightarrow \infty,$$

where $I(\theta_0, \theta_1, N) = (\theta_1 - \theta_0)^2 M / 4\theta_0$. (Also see Kutoyants 04)
To prove it, “Theory on Large Deviation” is needed.
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**Main Steps:**

- Calculate the Moment Generating Function of the Log-Likelihood ratio (Gapeev and Küchler [2008])
  - Use Feynman-Kac Formula to derive a PDE
  - Make some transforms and guess the solution
- Apply a theorem for Large Deviation in Lin’kov [1999]
- Use some technics in limit theory to get the final result
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- Calculate the Moment Generating Function of the Log-Likelihood ratio (Gapeev and Küchler [2008])
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  - Make some transforms and guess the solution

- Apply a theorem for Large Deviation in Lin’kov [1999]

- Use some technics in limit theory to get the final result

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Existing Literatures on Theory for This Hypothesis Testing Problem Stops Here!
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We already have:

- Under the Null Hypothesis $\mathcal{H}_0$:
  
  \[ (R^\#_T)_{T \in \mathbb{R}^+} \in \mathcal{K}_\alpha^*, \text{ and } \lim_{T \to \infty} P_{\theta_0}^{N,T}(R^\#_T) = \alpha, \text{ where } \]

  \[ \mathcal{K}_\alpha^* = \left\{ (R_T)_{T \in \mathbb{R}^+} : \limsup_{T \to \infty} \mathbb{P}_{\theta_0}^{N,T}(R_T) \leq \alpha \right\}. \]

- Under the Alternative $\mathcal{H}_1$:

  
  \[ 1 - \mathbb{P}_{\theta_1}^{N,T}(R^\#_T) \sim \exp(-I(\theta_0, \theta_1, N)T + o(T)). \]
We Go Further.
We Go Further.

Questions to be answered:

- Except for \( (\mathcal{R}_T^T) \), how do other rejection regions work for the testing? Is \( (\mathcal{R}_T^H) \) the best one?
- Is class \( \mathcal{K}_\alpha^* \) the best to take for the testing?
- How large \( T \) shall we take to insure the accuracy?
- Is Likelihood ratio type test the only way to deal with Simple Hypothesis?
We say that a rejection region \( (R_T^*) \in \mathcal{K}_\alpha^* \) is **asymptotically the most powerful** in the class \( \mathcal{K}_\alpha^* \) if

\[
\liminf_{T \to \infty} \frac{1 - \mathbb{P}_{\theta_1}^{N,T}(R_T)}{1 - \mathbb{P}_{\theta_1}^{N,T}(R_T^*)} \geq 1, \quad \text{for all } (R_T) \in \mathcal{K}_\alpha^*.
\] (3.5)

Similarly, we define asymptotically the most powerful rejection regions for a different given class of tests.
(Criterion for Most Powerful Test)
Consider the rejection region of the form

\[ R_T^* = \left\{ U_T^N : L(\theta_0, \theta_1, U_T^N) \geq c_{\alpha}^*(T) \right\}, \tag{3.6} \]

where \( c_{\alpha}^*(T) \) is a function of \( T \) such that, \( c_{\alpha}^*(T) > 0 \) for all \( T > 0 \) and

\[ \lim_{T \to \infty} \mathbb{P}_{\theta_0}^N(T)(R_T^*) = \alpha, \tag{3.7} \]

\[ \lim_{T \to \infty} \frac{c_{\alpha}^*(T)}{1 - \mathbb{P}_{\theta_1}^N(T)(R_T^*)} < \infty. \tag{3.8} \]

Then \( (R_T^*) \) is asymptotically the most powerful in \( \mathcal{K}_{\alpha}^* \).
However, to verify

$$\lim_{T \to \infty} \frac{c^*_\alpha(T)}{1 - \mathbb{P}_{\theta_1}^{N,T}(R^*_T)} < \infty$$

is not an easy task.
However, to verify
\[
\lim_{T \to \infty} \frac{c_\alpha^*(T)}{1 - P_{\theta_1}^{N,T}(R_T^*)} < \infty
\]
is not an easy task.

**We Need “Sharp Large Deviation”**

After a series of technical lemmas, we obtain
\[
c_\alpha^#(T) / \left(1 - P_{\theta_1}^{N,T}(R_T^#)\right) \sim \sqrt{T}, \quad \text{as } T \to \infty,
\]
where
\[
R_T^# := \{U_T^N : L(\theta_0, \theta_1, U_T^N) \geq c_\alpha^#(T)\}, \quad \text{for all } T,
\]
\[
c_\alpha^#(T) = \exp\left(-\frac{(\theta_1 - \theta_0)^2}{4\theta_0} MT - \frac{\theta_1^2 - \theta_0^2}{2\theta_0} \sqrt{\frac{MT}{2\theta_0}} q_\alpha\right).
\]
Problem Emerges:

The condition

\[
\lim_{T \to \infty} \frac{c^\#(T)}{1 - \mathbb{P}_{\theta_1}^{N,T}(R_T^\#)} < \infty
\]

is not satisfied!
Problem Emerges:
The condition
\[ \lim_{T \to \infty} \frac{c^\#(T)}{1 - \mathbb{P}_{\theta_1}^{N,T}(R^\#_T)} < \infty \]
is not satisfied!

How to Fix?
- Modify \( c^\#_\alpha \) to make \( (R^\#_T) \) asymptotically the most powerful.
Hypothesis Testing

Problem Emerges:
The condition

$$\lim_{T \to \infty} \frac{c_{\alpha}(T)}{1 - \mathbb{P}_{\theta_1}^{N,T}(R_T^\#)} < \infty$$

is not satisfied!

How to Fix?

- Modify $c_{\alpha}$ to make $(R_T^\#)$ asymptotically the most powerful.

However, this does not change the final result

$$\frac{c_{\alpha}(T)}{1 - \mathbb{P}_{\theta_1}^{N,T}(R_T^\#)} \sim \sqrt{T}.$$
Problem Emerges:

The condition

$$\lim_{T \to \infty} \frac{c^\#_\alpha(T)}{1 - \mathbb{P}_{\theta_1}^{N,T}(R^\#_T)} < \infty$$

is not satisfied!

How to Fix?

- Modify $c^\#_\alpha$ to make $(R^\#_T)$ asymptotically the most powerful. However, this does not change the final result

$$c^\#_\alpha(T) / \left(1 - \mathbb{P}_{\theta_1}^{N,T}(R^\#_T)\right) \sim \sqrt{T}.$$

- Recall that $\mathcal{K}^*_\alpha = \left\{ (R_T) : \limsup_{T \to \infty} \mathbb{P}_{\theta_0}^{N,T}(R_T) \leq \alpha \right\}$. 
Problem Emerges:

The condition

$$\lim_{T \to \infty} \frac{c^\#(T)}{1 - P^{N,T}_{\theta_1}(R^\#_T)} < \infty$$

is not satisfied!

How to Fix?

- Modify $c^\#_\alpha$ to make $(R^\#_T)$ asymptotically the most powerful. However, this does not change the final result

$$c^\#_\alpha(T)/\left(1 - P^{N,T}_{\theta_1}(R^\#_T)\right) \sim \sqrt{T}.$$

- Recall that $\mathcal{K}_\alpha^* = \left\{(R_T) : \limsup_{T \to \infty} P^{N,T}_{\theta_0}(R_T) \leq \alpha \right\}$.

Maybe we should redefine the Asymptotic Class.
Theorem (X., Igor Cialenco)

There exists rejection region \((\bar{R}_T)\) which is **Asymptotically More Powerful** than \((R^T)\), that is

\[
\limsup_{T \to \infty} \frac{1 - P^{N,T}_{\theta_1}(\bar{R}_T)}{1 - P^{N,T}_{\theta_1}(R^T)} < 1.
\]
Theorem (X., Igor Cialenco)

There exists rejection region \((\bar{R}_T)\) which is Asymptotically More Powerful than \((R_T^\#)\), that is

\[
\limsup_{T \to \infty} \frac{1 - P_{\theta_1}^{N,T}(\bar{R}_T)}{1 - P_{\theta_1}^{N,T}(R_T^\#)} < 1.
\]

The Most Powerful Test Does Not Exist In \(\mathcal{K}_{\alpha}^*\)!!

Theorem (X., Igor Cialenco)

The rejection region of the form

\[
R_T := \{ U_T^N : L(\theta_0, \theta_1, U_T^N) \geq c_{\alpha}(T) \}
\]

with \(c_{\alpha}(T) > 0\), can not be asymptotically the most powerful in the class \(\mathcal{K}_{\alpha}^*\).
Consider the class of the form:

\[ \mathcal{K}_\alpha^\# := \left\{ (R_T) : \limsup_{T \to \infty} \left( P_{\theta_0}^{N,T}(R_T) - \alpha \right) \sqrt{T} \leq \alpha_1 \right\}. \]
Consider the class of the form:

\[ \mathcal{K}^\#_\alpha := \left\{ (R_T) : \limsup_{T \to \infty} \left( \mathbb{P}^{N,T}_{\theta_0}(R_T) - \alpha \right) \sqrt{T} \leq \alpha_1 \right\}. \]

Then, what is \( \alpha_1 \)?
Lemma (X., Igor Cialenco)

For any integer \( n > 0 \) and \( x \in \mathbb{R} \), we have the following expansion

\[
\mathbb{P}_{\theta_0}^{N,T} (I_T \leq x) = \sum_{k=0}^{n} F_k(x) T^{-k/2} + \mathcal{R}_{n+1}^T(x) T^{-(n+1)/2}, \tag{3.9}
\]

where \( I_T = -\frac{\sqrt{8\theta_0^3}}{(\theta_1^2-\theta_0^2)\sqrt{T M}} \ln L(\theta_0, \theta_1, U_T^N) - \frac{(\theta_1-\theta_0)\sqrt{\theta_0 T M/2}}{\theta_1+\theta_0} \),

and \( F_k(\cdot) \) is a bounded smooth function and all its derivatives are bounded, and \( \mathcal{R}_{n+1}^T(\cdot) \) is uniformly bounded in \( x \).
Lemma (X., Igor Cialenco)

For any integer \( n > 0 \) and \( x \in \mathbb{R} \), we have the following expansion

\[
P_{\theta_0}^{N,T}(I_T \leq x) = \sum_{k=0}^{n} F_k(x) T^{-k/2} + \mathcal{R}_{n+1}^T(x) T^{-(n+1)/2},
\]

where \( I_T = -\frac{\sqrt{8\theta_0^3}}{(\theta_1^2 - \theta_0^2)\sqrt{T M}} \ln L(\theta_0, \theta_1, U_T^N) - \frac{(\theta_1 - \theta_0)\sqrt{\theta_0 T M / 2}}{\theta_1 + \theta_0}, \)

and \( F_k(\cdot) \) is a bounded smooth function and all its derivatives are bounded, and \( \mathcal{R}_{n+1}^T(\cdot) \) is uniformly bounded in \( x \).

Note that \( R_T^\# = \{ I_T \leq q_\alpha \} \), so by this lemma we have

\[
\lim_{T \to \infty} \left( P_{\theta_0}^{N,T}(R_T^\#) - \alpha \right) \sqrt{T} = F_1(q_\alpha).
\]
Lemma (X., Igor Cialenco)

For any integer \( n > 0 \) and \( x \in \mathbb{R} \), we have the following expansion

\[
\mathbb{P}_{\theta_0}^{N,T} (I_T \leq x) = \sum_{k=0}^{n} F_k(x) T^{-k/2} + \mathcal{R}_{n+1}^{T}(x) T^{-(n+1)/2},
\]

where \( I_T = -\frac{\sqrt{8\theta_0^3}}{(\theta_1^2 - \theta_0^2)\sqrt{T}} \ln L(\theta_0, \theta_1, U_T^N) - \frac{(\theta_1 - \theta_0)\sqrt{\theta_0TM/2}}{\theta_1 + \theta_0} \),

and \( F_k(\cdot) \) is a bounded smooth function and all its derivatives are bounded, and \( \mathcal{R}_{n+1}(\cdot) \) is uniformly bounded in \( x \).

Note that \( R_T^\# = \{I_T \leq q_\alpha\} \), so by this lemma we have

\[
\lim_{T \to \infty} \left( \mathbb{P}_{\theta_0}^{N,T}(R_T^\#) - \alpha \right) \sqrt{T} = F_1(q_\alpha).
\]

Therefore, it is reasonable to take

\[
\alpha_1 = F_1(q_\alpha)
\]
Main Result (I):

Theorem (X., Igor Cialenco)

(The Most Powerful Test) The rejection region $\left( R^T \right)$ is asymptotically the most powerful in the class $\mathcal{K}^\#_{\alpha}$ with $\alpha_1 = F_1(q_{\alpha})$. 
Define class

\[ \tilde{\mathcal{K}}_\alpha := \left\{ (R_N)_{N \in \mathbb{N}^+} : \limsup_{N \to \infty} \mathbb{P}_{\theta_0}^N \left( R_N \right) \leq \alpha \right\}. \]

**Definition**

We say that a rejection region \((\tilde{R}_N) \in \tilde{\mathcal{K}}_\alpha\) is **asymptotically the most powerful** in the class \(\tilde{\mathcal{K}}_\alpha\) if

\[
\liminf_{N \to \infty} \frac{1 - \mathbb{P}_{\theta_1}^N \left( R_N \right)}{1 - \mathbb{P}_{\theta_1}^N \left( \tilde{R}_N \right)} \geq 1, \quad \text{for all } (R_N) \in \tilde{\mathcal{K}}_\alpha. \quad (3.10)
\]

Similarly, we define asymptotically the most powerful rejection regions for a different given class of tests.
Theorem (X., Igor Cialenco)

(Criterion for Most Powerful Test)

Consider the rejection region of the form

\[ \tilde{R}_N = \{ U_T^N : L(\theta_0, \theta_1, U_T^N) \geq \tilde{c}_\alpha(T) \} , \quad (3.11) \]

where \( \tilde{c}_\alpha(T) \) is a function of \( T \) such that, \( \tilde{c}_\alpha(T) > 0 \) for all \( T > 0 \) and

\[ \lim_{N \to \infty} \mathbb{P}_{\theta_0}^{N,T}(\tilde{R}_N) = \alpha, \quad (3.12) \]

\[ \lim_{N \to \infty} \frac{\tilde{c}_\alpha(N)}{1 - \mathbb{P}_{\theta_1}^{N,T}(\tilde{R}_N)} < \infty, \quad (3.13) \]

then \( (\tilde{R}_N) \) is asymptotically the most powerful in \( \tilde{K}_\alpha \).
Following similar argument as in “T part”, one can find

\[ \hat{R}_N = \left\{ U_T^N : L(\theta_0, \theta_1, U_T^N) \geq \hat{c}_\alpha (N) \right\}, \]

with

\[ \hat{c}_\alpha (N) = \exp \left( -\frac{(\theta_1 - \theta_0)^2 T M}{4 \theta_0} + \frac{(\theta_1 - \theta_0)^2 N}{8 \theta_0^2} - \frac{\sqrt{T M} (\theta_1^2 - \theta_0^2)}{\sqrt{8 \theta_0^3}} q_\alpha \right), \]

and prove that

\[ \lim_{N \to \infty} \mathbb{P}_{\theta_0}^{N,T} (\hat{R}_N) = \alpha. \]

By using the same strategy as in “T part”, one can also show

\[ \hat{c}_\alpha (N) / \left( 1 - \mathbb{P}_{\theta_1}^{N,T} (\hat{R}_N) \right) \sim \sqrt{M} \sim N^{\beta/d+1/2}. \]
Theorem (X., Igor Cialenco)

There exists rejection region \((\bar{R}_N)\) which is **Asymptotically More Powerful** than \((\hat{R}_N)\), that is

\[
\limsup_{T \to \infty} \frac{1 - P_{\theta_1}^{N,T}(\bar{R}_N)}{1 - P_{\theta_1}^{N,T}(\hat{R}_N)} < 1.
\]

Theorem (X., Igor Cialenco)

The rejection region of the form

\[
R_N := \{ U_T^N : L(\theta_0, \theta_1, U_T^N) \geq c_{\alpha}(N) \},
\]

with \(c_{\alpha}(N) > 0\), can not be asymptotically the most powerful in the class \(\tilde{K}_\alpha\).
Lemma (X., Igor Cialenco)

For any $x \in \mathbb{R}$, we have the following expansion,

$$P_{\theta_0}^{N,T} (I^N \leq x) = \Phi(x) + \Phi_1^\delta(x) M^{-1/2} + \Phi_2^\delta(x) N M^{-1}$$

$$+ \mathcal{R}_N^\delta(x) \left( M^{-1} + N M^{-3/2} + N^2 M^{-2} \right), \quad (3.14)$$

where

$$I^N = -\frac{\sqrt{8\theta_0^3 \ln L(\theta_0, \theta_1, U_T^N)}}{\sqrt{T M (\theta_1^2 - \theta_0^2)}} - \frac{\sqrt{2\theta_0 T M (\theta_1 - \theta_0)}}{2(\theta_1 + \theta_0)} + \frac{(\theta_1 - \theta_0) N}{\sqrt{8\theta_0 T M (\theta_1 + \theta_0)}},$$

and $\Phi(\cdot)$ is the distribution of a standard Gaussian random variable, and $\Phi_1(\cdot)$ and $\Phi_2(\cdot)$ are bounded smooth functions and all their derivatives are bounded, and $\mathcal{R}_N(\cdot)$ is uniformly bounded in $x$. 
Define

\[ \hat{K}_\alpha := \left\{ (R_N) : \limsup_{N \to \infty} \left( \mathbb{P}_{\theta_0}^{N,T}(R_N) - \alpha \right) \sqrt{M} \leq \hat{\alpha}_1 \right\} . \] (3.15)

Note that \( \hat{R}_N = \{ U_N^T : I_N^N \leq q_\alpha \} \), so by the previous lemma we have

\[ \lim_{N \to \infty} \left( \mathbb{P}_{\theta_0}^{N,T}(\hat{R}_N) - \alpha \right) \sqrt{M} = \hat{\alpha}_1 , \]

where

\[ \hat{\alpha}_1 = \begin{cases} \Phi_1(q_\alpha) , & \text{if } \beta/d > 1/2 \\ \Phi_1(q_\alpha) + \sqrt{\frac{2\beta/d + 1}{\omega}} \Phi_2(q_\alpha) , & \text{if } \beta/d = 1/2 \end{cases} . \]
Define

\[ \hat{K}_\alpha := \left\{ (R_N) : \limsup_{N \to \infty} \left( \mathbb{P}_{\theta_0}^{N,T}(R_N) - \alpha \right) \sqrt{M} \leq \hat{\alpha}_1 \right\} . \] (3.15)

Note that \( \hat{R}_N = \{ U^N_T : I^N \leq q_\alpha \} \), so by the previous lemma we have

\[ \lim_{N \to \infty} \left( \mathbb{P}_{\theta_0}^{N,T}(\hat{R}_N) - \alpha \right) \sqrt{M} = \hat{\alpha}_1, \]

where

\[ \hat{\alpha}_1 = \begin{cases} \Phi_1(q_\alpha), & \text{if } \beta/d > 1/2 \\ \Phi_1(q_\alpha) + \sqrt{\frac{2\beta/d + 1}{\omega^\beta}} \Phi_2(q_\alpha), & \text{if } \beta/d = 1/2 \end{cases} . \]

**Theorem (Main Result II)**

Assume \( \beta/d \geq 1/2 \). The rejection region \((\hat{R}_N)\) is asymptotically the most powerful in \( \hat{K}_\alpha \).
Thank You!
Theorem (Criterion for Most Powerful Test)

Consider the rejection region of the form

\[ R_T^* = \left\{ U_T^N : L(\theta_0, \theta_1, U_T^N) \geq c_{\alpha}(T) \right\}, \quad (3.16) \]

where \( c_{\alpha}(T) \) is a function of \( T \) such that, \( c_{\alpha}(T) > 0 \) for all \( T > 0 \) and

\[
\lim_{T \to \infty} \frac{c_{\alpha}(T)}{1 - \mathbb{P}_{\theta_1}^{N,T}(R_T^*)} < \infty. \quad (3.18)
\]

Then \((R_T^*)\) is asymptotically the most powerful in \( \mathcal{K}_{\alpha}^* \).
Proof:
By the same reasoning as in "Neyman-Pearson", for a fixed $T$ and any $(R_T) \in K^*_\alpha$, we have that

$$\mathbb{P}_{\theta_1}^{N,T}(R^*_T) - \mathbb{P}_{\theta_1}^{N,T}(R_T) \geq c^*_\alpha(T) \left( \mathbb{P}_{\theta_0}^{N,T}(R^*_T) - \mathbb{P}_{\theta_0}^{N,T}(R_T) \right),$$
Proof:
By the same reasoning as in "Neyman-Pearson", for a fixed $T$ and any $(R_T) \in \mathcal{K}_{\alpha}^{*}$, we have that

$$\mathbb{P}_{\theta_1}^{N,T}(R_T) - \mathbb{P}_{\theta_1}^{N,T}(R_T) \geq c_{\alpha}^{*}(T) \left( \mathbb{P}_{\theta_0}^{N,T}(R_T^*) - \mathbb{P}_{\theta_0}^{N,T}(R_T) \right),$$

which can be written as

$$\frac{1 - \mathbb{P}_{\theta_1}^{N,T}(R_T)}{1 - \mathbb{P}_{\theta_1}^{N,T}(R_T^*)} \geq 1 + \frac{c_{\alpha}^{*}(T)}{1 - \mathbb{P}_{\theta_1}^{N,T}(R_T^*)} \left( \mathbb{P}_{\theta_0}^{N,T}(R_T^*) - \mathbb{P}_{\theta_0}^{N,T}(R_T) \right).$$
From here, using (3.17) and (3.18), we deduce

\[
\lim \inf_{T \to \infty} \frac{1 - \mathbb{P}_{\theta_1}^{N,T}(R_T)}{1 - \mathbb{P}_{\theta_1}^{N,T}(R_T^*)} \geq 1 + \lim_{T \to \infty} \frac{c^*_\alpha(T)}{1 - \mathbb{P}_{\theta_1}^{N,T}(R_T^*)} \lim_{T \to \infty} \mathbb{P}_{\theta_0}^{N,T}(R_T^*) \\
- \lim_{T \to \infty} \frac{c^*_\alpha(T)}{1 - \mathbb{P}_{\theta_1}^{N,T}(R_T^*)} \limsup_{T \to \infty} \mathbb{P}_{\theta_0}^{N,T}(R_T) \\
= 1 + \lim_{T \to \infty} \frac{c^*_\alpha(T)}{1 - \mathbb{P}_{\theta_1}^{N,T}(R_T^*)} \left( \alpha - \limsup_{T \to \infty} \mathbb{P}_{\theta_0}^{N,T}(R_T) \right) \\
\geq 1.
\]

This completes the proof.
Proof for $c^\#_\alpha(T) / \left(1 - \mathbb{P}^{N,T}_{\theta_1}(R^\#_T)\right) \sim \sqrt{T}$:

- Split the probability:

$$1 - \mathbb{P}^{N,T}_{\theta_1}(R^\#_T) = A_T B_T$$

- After some substitutions and sophisticated calculation we get

$$A_T \approx \exp[-I(\theta_0, \theta_1, N)T]$$

- By a series of technical lemmas we proved

$$B_T \sim \exp[o(T)] / \sqrt{T}$$

- Referring to the form of $c^\#_\alpha$ in (3.4) we finally have

$$c^\#_\alpha(T) / \left(1 - \mathbb{P}^{N,T}_{\theta_1}(R^\#_T)\right) \sim \sqrt{T}$$
Sketch of the proof for main theorem:
By the same reasoning as in "Neyman-Pearson", for a fixed $T$ and any $(R_T) \in K^*_\alpha$, we have that

$$
\mathbb{P}^{N,T}_{\theta_1}(R_T^\#) - \mathbb{P}^{N,T}_{\theta_1}(R_T) \geq c^\#_\alpha(T) \left( \mathbb{P}^{N,T}_{\theta_0}(R_T^\#) - \mathbb{P}^{N,T}_{\theta_0}(R_T) \right),
$$

which can be written as

$$
\frac{1 - \mathbb{P}^{N,T}_{\theta_1}(R_T)}{1 - \mathbb{P}^{N,T}_{\theta_1}(R_T^\#)} \geq 1 + \frac{c^\#_\alpha(T)}{1 - \mathbb{P}^{N,T}_{\theta_1}(R_T^\#)} \left( \mathbb{P}^{N,T}_{\theta_0}(R_T^\#) - \mathbb{P}^{N,T}_{\theta_0}(R_T) \right).
$$

Taking the 'liminf', we deduce

$$
\liminf_{T \to \infty} \frac{1 - \mathbb{P}^{N,T}_{\theta_1}(R_T)}{1 - \mathbb{P}^{N,T}_{\theta_1}(R_T^\#)} \geq 1 + \liminf_{T \to \infty} \frac{c^\#_\alpha(T)}{1 - \mathbb{P}^{N,T}_{\theta_1}(R_T^\#)} \left( \mathbb{P}^{N,T}_{\theta_0}(R_T^\#) - \mathbb{P}^{N,T}_{\theta_0}(R_T) - \alpha \right)
$$

$$
- \limsup_{T \to \infty} \frac{c^\#_\alpha(T)}{1 - \mathbb{P}^{N,T}_{\theta_1}(R_T^\#)} \left( \mathbb{P}^{N,T}_{\theta_0}(R_T) - \alpha \right).
$$