1. Consider the transformation that rotates each vector $x$ in $\mathbb{R}^2$ by angle $\theta$ in the counterclockwise direction.

(a) Show that the transformation is linear.

(b) Find a matrix $Q$ that represents the transformation.

(c) Show that $Q$ is an orthogonal matrix.

(d) Show that $\|Qx\|_2 = \|x\|_2$. Interpret the formula.

(e) Find $Q^{-1}$ and describe how it transforms a vector $x$ in $\mathbb{R}^2$.

2. Let $S$ be a subspace of $\mathbb{R}^3$ spanned by $\{(2/3, 2/3, 1/3)^T, (1/\sqrt{2}, -1/\sqrt{2}, 0)^T\}$. Let $x = (1, 1, 2)^T$. Find the projection vector $p$ of $x$ onto $S$. Show that $(p - x)$ is perpendicular to the two spanning vectors of $S$.

3. Let $\{u_1, u_2, u_3\}$ be an orthonormal basis for the real inner product space $V$. If $x = c_1u_1 + c_2u_2 + c_3u_3$ such that $\|x\| = 5$ (defined by the standard inner), $u_1^T \cdot x = 4$, and $x \perp u_2$. Find $c_1, c_2, c_3$.

4. Show $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ is a norm in $\mathbb{R}^n$.

5. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$ 

Prove using a first principle argument (geometric) by mapping the unit circle $\|x\|_2 = 1$ into its image under $A$ that $\|A\|_2 \approx 1.6180$, as described in class.

6. Find the IEEE double precision presentation $fl(x)$ and the rounding error for $x = 0.4$.

7. Find the roots of $x^2 + 3x - 8^{-14} = 0$ using the quadratic formula using 3 digit arithmetic. Give an improved algorithm for the computing the roots and demonstrate it using 4 digit arithmetic.