1. (a) Show that the functions \( \phi_1(x) = 1 \) and \( \phi_2(x) = x \) are orthogonal on the interval \([-1, 1]\) with respect to the weight function \( \sigma(x) = 1 \).

(b) Determine the constants \( a \) and \( b \) such that a third function \( \phi_3 \) of the form

\[
\phi_3(x) = ax^2 + bx + 1
\]

is orthogonal to both \( \phi_1 \) and \( \phi_2 \).

2. When we apply separation of variables and a substitution \( x = \cos \theta \) to Laplace’s equation in spherical coordinates we obtain a differential equation of the form

\[
\frac{d}{dx} \left[ (1 - x^2)\phi'(x) \right] + \lambda \phi(x) = 0.
\]

This equation is known as Legendre’s equation.

(a) What conditions need to be added to make this a regular Sturm-Liouville problem?

(b) One can show that the eigenvalues of Legendre’s equation are \( \lambda_n = (n - 1)n, \ n = 1, 2, 3, \ldots \). Verify that the functions \( \phi_n, \ n = 1, 2, 3, \) of the previous problem are the associated eigenfunctions.

(c) What is the formula for the generalized Fourier coefficients \( a_n \) in the eigenfunction expansion of \( f(x) = \sin \pi x \)? Compute the first three of them.

3. The problem

\[
A(x)\phi''(x) + B(x)\phi'(x) + C(x)\phi(x) + \lambda D(x)\phi(x) = 0
\]

is in standard Sturm-Liouville form only if \( B(x) = A'(x) \).

(a) Show that if \( A(x) \neq 0 \) on \([a, b]\) and \( \frac{B - A'}{A} \) is continuous on \([a, b]\) then multiplication of (1) by

\[
w(x) = e^{\int \frac{B(x) - A'(x)}{A(x)} \, dx}
\]

will lead to standard Sturm-Liouville form.

(b) Use the idea of (a) to convert \( \phi''(x) + \phi'(x) + \lambda \phi(x) = 0 \) to standard Sturm-Liouville form.

(c) Use the idea of (a) to convert \( x^2\phi''(x) + x\phi'(x) + \lambda \phi(x) = 0 \) to standard Sturm-Liouville form.

4. Consider the eigenvalue problem

\[
x^2\phi''(x) + x\phi'(x) + \lambda \phi(x) = 0, \quad 1 < x < a,
\]

\[
\phi(1) = 0, \quad \phi(a) = 0.
\]

(a) Show that the eigenvalues and eigenfunctions are given by

\[
\lambda_n = \left( \frac{n\pi}{\ln a} \right)^2, \quad \text{and} \quad \phi_n(x) = \sin \left( \frac{n\pi \ln x}{\ln a} \right).
\]

Hint: Recognize the differential equation as one of Cauchy-Euler type.
(b) Show that the eigenfunction expansion of a given piecewise smooth function \( f \) is of the form

\[
 f(x) = \sum_{n=1}^{\infty} c_n \sin \left( n\pi \frac{\ln x}{\ln a} \right),
\]

where

\[
 c_n = \frac{\int_1^a f(x) \sin \left( n\pi \frac{\ln x}{\ln a} \right) \, dx}{\int_1^a \sin^2 \left( n\pi \frac{\ln x}{\ln a} \right) \, dx}.
\]

Hint: First convert the eigenvalue problem to standard Sturm-Liouville form as in the previous problem. This will tell you what the weight function \( \sigma \) is.

5. The following are usually assigned as HW problems (and are recommended for you to look at):

(a) Problem 5.3.5 in the textbook.
(b) Problem 5.4.2 in the textbook.
(c) Problem 5.4.3 in the textbook.