3 Projectors

If $P \in \mathbb{C}^{m \times m}$ is a square matrix such that $P^2 = P$ then $P$ is called a projector. A matrix satisfying this property is also known as an idempotent matrix.

**Remark** It should be emphasized that $P$ need not be an orthogonal projection matrix. Moreover, $P$ is usually not an orthogonal matrix.

**Example** Consider the matrix

$$P = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix},$$

where $c = \cos \theta$ and $s = \sin \theta$. This matrix projects perpendicularly onto the line with inclination angle $\theta$ in $\mathbb{R}^2$.

We can check that $P$ is indeed a projector:

$$P^2 = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} = \begin{bmatrix} c^4 + c^2s^2 & c^3s + cs^3 \\ c^3s + cs^3 & c^2s^2 + s^4 \end{bmatrix} = \begin{bmatrix} c^2(c^2 + s^2) & cs(c^2 + s^2) \\ cs(c^2 + s^2) & s^2(c^2 + s^2) \end{bmatrix} = P.$$

Note that $P$ is not an orthogonal matrix, i.e., $P^*P = P^2 = P \neq I$. In fact, $\text{rank}(P) = 1$ since points on the line are projected onto themselves.

**Example** The matrix

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

is clearly a projector. Since the range of $P$ is given by all points on the $x$-axis, and any point $(x, y)$ is projected to $(x + y, 0)$, this is clearly not an orthogonal projection.

In general, for any projector $P$, any $v \in \text{range}(P)$ is projected onto itself, i.e., $v = Px$ for some $x$ then

$$Pv = P(Px) = P^2x = Px = v.$$

We also have

$$P(Pv - v) = P^2v - Pv = Pv - Pv = 0,$$

so that $Pv - v \in \text{null}(P)$.

### 3.1 Complementary Projectors

In fact, $I - P$ is known as the complemental projector to $P$. It is indeed a projector since

$$(I - P)^2 = (I - P)(I - P) = I - IP - PI + P^2 = I - P.$$
Lemma 3.1 If \( P \) is a projector then

\[
\begin{align*}
\text{range}(I - P) &= \text{null}(P), \\
\text{null}(I - P) &= \text{range}(P).
\end{align*}
\] (10)

\[
\begin{align*}
\text{null}(I - P) &= \text{range}(P).
\end{align*}
\] (11)

Proof We show (10), then (11) will follow by applying the same arguments for \( P = I - (I - P) \). Equality of two sets is shown by mutual inclusions, i.e., \( A = B \) if \( A \subseteq B \) and \( B \subseteq A \).

First, we show \( \text{null}(P) \subseteq \text{range}(I - P) \). Take a vector \( v \) such that \( Pv = 0 \). Then \( (I - P)v = v - Pv = v \). In words, any \( v \) in the nullspace of \( P \) is also in the range of \( I - P \).

Now, we show \( \text{range}(I - P) \subseteq \text{null}(P) \). We know that any \( x \in \text{range}(I - P) \) is characterized by

\[
\begin{align*}
x &= (I - P)v \quad \text{for some } v.
\end{align*}
\]

Thus

\[
\begin{align*}
x &= v - Pv = -(Pv - v) \in \text{null}(P)
\end{align*}
\]

since we showed earlier that \( P(Pv - v) = 0 \). Thus if \( x \in \text{range}(I - P) \), then \( x \in \text{null}(P) \).

\[\square\]

3.2 Decomposition of a Given Vector

Using a projector and its complementary projector we can decompose any vector \( v \) into

\[
v = Pv + (I - P)v,
\]

where \( Pv \in \text{range}(P) \) and \( (I - P)v \in \text{null}(P) \). This decomposition is unique since \( \text{range}(P) \cap \text{null}(P) = \{0\} \), i.e., the projectors are complementary.

3.3 Orthogonal Projectors

If \( P \in \mathbb{C}^{m \times m} \) is a square matrix such that \( P^2 = P \) and \( P = P^* \) then \( P \) is called an orthogonal projector.

Remark In some books the definition of a projector already includes orthogonality. However, as before, \( P \) is in general not an orthogonal matrix, i.e., \( P^*P = P^2 \neq I \).

3.4 Connection to Earlier Orthogonal Decomposition

Earlier we considered the orthonormal set \( \{q_1, \ldots, q_n\} \), and established the decomposition

\[
\begin{align*}
v &= r + \sum_{i=1}^{n} (q_i^*v)q_i \\
&= r + \sum_{i=1}^{n} (q_iq_i^*)v
\end{align*}
\] (12)
with \( r \) orthogonal to \( \{ q_1, \ldots, q_n \} \). This corresponds to the decomposition

\[ v = (I - P)v + Pv \]

with \( P = \sum_{i=1}^{n} (q_iq_i^*) \).

Note that \( \sum_{i=1}^{n} (q_iq_i^*) = QQ^* \) with \( Q = [q_1 q_2 \cdots q_n] \). Thus the orthogonal decomposition (12) can be rewritten as

\[ v = (I - QQ^*)v + QQ^*v. \] (13)

It is easy to verify that \( QQ^* \) is indeed an orthogonal projection:

1. \( (QQ^*)^2 = QQ^*QQ^* = QQ^* \) since \( Q \) has orthonormal columns (but not rows).
2. \( (QQ^*)^* = QQ^* \).

**Remark** The orthogonal decomposition (13) will be important for the implementation of the QR decomposition later on. In particular we will use the rank-1 projector

\( P_q = qq^* \)

which projects onto the direction \( q \) and its complement

\( P_{\perp q} = I - qq^* \).

Thus,

\[ v = (I - qq^*)v + qq^*v, \]

or, more generally, orthogonal projections onto an arbitrary direction \( a \) is given by

\[ v = \left( I - \frac{aa^*}{a^*a} \right) v + \frac{aa^*}{a^*a} v, \]

where we abbreviate \( P_a = \frac{aa^*}{a^*a} \) and \( P_{\perp a} = (I - \frac{aa^*}{a^*a}) \).

As a further generalization we can consider orthogonal projection onto the range of a (full-rank) matrix \( A \). Earlier, for the orthonormal basis \( \{ q_1, \ldots, q_n \} \) (the columns of \( Q \)) we had \( P = QQ^* \). Now we require only that \( \{ a_1, \ldots, a_n \} \) be linearly independent. In order to compute the projection \( P \) for this case we start with an arbitrary vector \( v \).

We need to ensure that \( Pv - v \perp \text{range}(A) \), i.e., if \( Pv \in \text{range}(A) \) then

\[ a_j^*(Pv - v) = 0, \quad j = 1, \ldots, n. \]

Now, since \( Pv \in \text{range}(A) \) we know \( Pv = Ax \) for some \( x \). Thus

\[ a_j^*(Ax - v) = 0, \quad j = 1, \ldots, n \]

\[ A^*(Ax - v) = 0 \]
or

\[ A^*Ax = A^*v. \]

One can show that \((A^*A)^{-1}\) exists provided the columns of \(A\) are linearly independent (our assumption). Then

\[ x = (A^*A)^{-1}A^*v. \]

Finally,

\[ Pv = Ax = A(A^*A)^{-1}A^*v. \]

**Remark** Note that this includes the earlier discussion when \(\{a_1, \ldots, a_n\}\) is orthonormal since then \(A^*A = I\) and \(P = AA^*\) as before.