MATH 100 – Introduction to the Profession
The Need for Approximation

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Outline

1 Why Bother With Approximations?
2 Pure Math
3 Computational Math
4 Theoretical Computer Science
5 Mathematical Modeling

Part of this comes from [T. Gowers: Mathematics: A Very Short Introduction, Chapter 7], while some other topics may relate back to phenomena seen earlier.
Most people think of mathematics as a very clean, exact subject. One learns at school to expect ... a simple formula. Those who continue with mathematics at university level ... soon discover that nothing could be further from the truth. For many problems it would be miraculous and totally unexpected if somebody were to find a precise formula for the solution; most of the time one must settle for a rough estimate instead. Until one is used to estimates, they seem ugly and unsatisfying. However, it is worth acquiring a taste for them, because not to do so is to miss out on many of the greatest theorems and most interesting unsolved problems in mathematics.

Knowledge of a (rough) estimate or an approximate answer is important in many areas of math:

- **Pure math**: many results in number theory are proved using models and estimates based on probabilistic arguments (see, e.g., the prime number theorem below).
- **Computational math**: absolutely crucial in order to know how accurate, how reliable and how fast numerical algorithms are (see, e.g., Cramer’s rule and derivative approximation below).
- **Theoretical computer science**: similarly important to estimating the run-time of algorithms (see, e.g., the famous \( P \) vs. \( NP \) problem below).
- **Math modeling**: often so-called asymptotic analysis is used.
Prime Numbers

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Problem: We don’t really understand prime numbers and their distribution – and yet they have vast practical applications, especially in security algorithms (cryptography, RSA encoding, . . .). Possibly the biggest unsolved math problem: the Riemann hypothesis.
Approximate Information for Primes

Gauss investigated the distribution of prime numbers. He defined $\pi(N)$, the number of primes up to $N$, and conjectured

$$\pi(N) \approx \frac{1}{\ln 2} + \frac{1}{\ln 3} + \frac{1}{\ln 4} + \ldots + \frac{1}{\ln N}$$

based on his observation that the density of prime numbers is related to logarithms, i.e.,

$$\text{density} = \frac{\text{mass}}{\text{volume}} = \frac{\pi(N)}{N} = \frac{\text{number of primes}}{\text{length of interval}} \approx \frac{1}{\ln N} = \frac{1}{\text{average gap}}.$$
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See the MATLAB function `PrimeTheorem(N)` for this and a few other examples, and Marcus Du Sautoy’s `music of the primes` for more on the Riemann hypothesis.
Twin Primes Conjecture & Bounded Gaps

Even though the prime number theorem tells us that average gaps between primes are increasing, we also have

Twin primes: a pair of primes separated by a gap of 2, e.g.,

\[(3, 5), (5, 7), (11, 13), (17, 19), \ldots\]
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Approximating Derivatives

By dropping the limit from the definition of the derivative we can approximate the value $y'(t)$ of some function $y$ by using a forward difference approximation

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So, how accurate is it for more general functions?
Using a **Taylor series expansion** (see MATH 152) we can show that

\[ y(t + h) = y(t) + hy'(t) + \frac{h^2}{2} y''(\tau), \]

where \( \tau \) is somewhere between \( t \) and \( t + h \).
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This yields

\[ y'(t) = \frac{y(t + h) - y(t)}{h} - \frac{h}{2} y''(\tau), \]

so that the truncation error of the forward difference approximation depends on

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Clearly, we get better and better approximations by simply making \( h \) smaller$^2$.

$^2$Other derivative approximation methods give “more bang for the buck” by having a truncation error that goes to zero like \( h^2 \) as \( h \to 0 \) (see MATH 350).
How do we estimate the cost of Cramer’s rule?

Cramer’s rule states that the solution \( \mathbf{x} = (x_1, \ldots, x_n)^T \) of the linear system \( A\mathbf{x} = \mathbf{b} \) is given by

\[
x_i = \frac{\det A_i}{\det A}, \quad i = 1, \ldots, n,
\]

where \( A_i \) is obtained from \( A \) by replacing its \( i^{\text{th}} \) column by \( \mathbf{b} \). Therefore, we need to compute \( n + 1 \) determinants, which can be shown to require approximately \( 3n! \) arithmetic operations each\(^3\).

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<table>
<thead>
<tr>
<th>( n )</th>
<th>( 10^9 ) (Giga)</th>
<th>( 10^{10} )</th>
<th>( 10^{11} )</th>
<th>( 10^{12} ) (Tera)</th>
<th>( 10^{15} ) (Peta)</th>
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<tr>
<td>10</td>
<td>( 10^{-1} ) sec</td>
<td>( 10^{-2} ) sec</td>
<td>( 10^{-3} ) sec</td>
<td>( 10^{-4} ) sec</td>
<td>negligible</td>
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<tr>
<td>15</td>
<td>17 hours</td>
<td>1.74 hours</td>
<td>10.46 min</td>
<td>1 min</td>
<td>0.6 ( 10^{-1} ) sec</td>
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<tr>
<td>20</td>
<td>4860 years</td>
<td>486 years</td>
<td>48.6 years</td>
<td>4.86 years</td>
<td>1.7 day</td>
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<tr>
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<td>o.r.</td>
<td>o.r.</td>
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Theoretical Computer Science

The *P* vs. *NP* Problem

One task of theoretical computer science is to determine the complexity of algorithms. As we saw with Cramer’s rule (or with the recursive Fibonacci algorithm earlier), it is often not feasible to run an algorithm with very large input size, and so runtime must be estimated.
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In other words, is the class **P** (of “easily solvable” problems) the same as the class **NP** (of “easily checkable” problems)? See Ian Stewart’s excellent [analysis of the game Minesweeper](http://www.claymath.org/millenium-problems/p-vs-np-problem).
Asymptotic or Perturbation Analysis

In modeling situations, one strategy is to **write a problem in dimensionless form, and then analyze what happens for a parameter (known to be either small or large)**.
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Example (from [S. Howison: Practical Applied Mathematics])
Consider the quadratic equation (with small $\varepsilon$)

$$\varepsilon x^2 + x - 1 = 0$$

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Example (from [S. Howison: Practical Applied Mathematics])
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and find a solution without using the quadratic formula.

Idea: Consider the desired solution as a perturbation of the much simpler problem

$$x - 1 = 0,$$

which arises for $\varepsilon = 0$. 
To get an approximate solution for the original (perturbed) problem we make an Ansatz

\[ x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots \]
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Now substitute this into the original equation:

\[ \varepsilon (1 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots)^2 + (1 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots) - 1 = 0 \]
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This gives (from the Ansatz)
\[ x = 1 - \varepsilon + 2\varepsilon^2 + \ldots \]

as approximate solution to the original quadratic equation.
“Real” use of asymptotic analysis is called for in applications such as
- orbit calculations in astronomy,
- stability analysis of differential equations,
- boundary layers of differential equations as arise in fluid flow problems (e.g., water waves),
- modeling of lubricants,
- and many others.

See MATH 486 for more.
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