This file containing interesting problems from outside the textbook will be updated throughout the semester. Most of these problems will be assigned in the homework.

1. Use the Well-Ordering Principle to show that $\sqrt{2}$ is irrational.

2. Prove that the expression $(3^{3n+3} - 26n - 27)$ is a multiple of 169 for all $n \in \mathbb{N}$.

3. Let $x \neq 1$ be any real number. Then prove that $\sum_{j=0}^{n-1} x^j = \frac{x^n - 1}{x - 1}$.

4. Prove that if $a$ and $b$ are odd integers, then $a^2 - b^2$ is divisible by 8.

5. Prove that for all $n \in \mathbb{N}$, $(1 + \sqrt{2})^{2n} + (1 - \sqrt{2})^{2n}$ is an even integer and $(1 + \sqrt{2})^{2n} - (1 - \sqrt{2})^{2n}$ equals $b\sqrt{2}$ for some integer $b$.

(Hint: Prove both the statements simultaneously using induction on $n$.)

6. Prove that any two consecutive Fibonacci numbers are relatively prime.

(Fibonacci numbers are defined as: $f_0 = 1$, $f_1 = 1$, $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$.)

7. Use the textbook exercise 2.3.20f (solved in HW#2) to show that $\gcd(a^2, b^2) = (\gcd(a, b))^2$.

8. Let $a, b, c, d$ be positive integers with $b \neq d$. Prove that: If $\gcd(a, b) = 1$ and $\gcd(c, d) = 1$ then $\frac{a}{b} + \frac{c}{d}$ is not an integer.

9. Find the smallest positive integer $n$ such that the Diophantine equation $10x + 11y = n$ has exactly nine non-negative solutions.

10. What is the smallest positive rational number that can be expressed in the form $\frac{x}{50} + \frac{y}{36}$ with $x, y \in \mathbb{Z}$?

11. Prove that: If $2^n - 1$ is prime then $n$ is prime. [Compare this to #3.1.11b in the textbook.]

12. Let $F_n = 2^{2^n} + 1$, $n \geq 0$ (these are called Fermat numbers). Show that $\gcd(F_m, F_n) = 1$ for $m > n \geq 0$.

13. Prove there are infinitely many primes of the form $4k + 1$.

14. Find the smallest integer divisible by 2 and 3 which is simultaneously a square and a fifth power.
15. Prove that
\[ \frac{(\text{lcm}(a, b, c))^2}{\text{lcm}(a, b) \text{lcm}(b, c) \text{lcm}(c, a)} = \frac{(\text{gcd}(a, b, c))^2}{\text{gcd}(a, b) \text{gcd}(b, c) \text{gcd}(c, a)} \]

16. If $p$ and $p + 2$ are twin primes, with $p > 3$, then prove that $6 \mid (p + 1)$.

17. Let $N = (a_m a_{m-1} \ldots a_2 a_1 a_0)_{10}$. Let $M = a_m 10^{m-1} + \ldots + a_2 10^2 + a_1 10^1 + a_1$. Then show that
   (a) $7 \mid N \iff 7 \mid (M - 2a_0)$.
   (b) $13 \mid N \iff 13 \mid (M - 9a_0)$.
   [Note that the repeated application of these criteria on a number gives an efficient procedure to check for divisibility by 7 or 13.]

18. Find one million consecutive integers that are NOT square-free. [Note that $n$ is not square-free iff $p^2 \mid n$ for some prime $p$, i.e., square (or a higher power) of a prime occurs in its prime factorization.]

19. The converse to the Fermat’s little theorem is true in the following sense:
   Show that: If $n \geq 2$ and for all $a$, $1 \leq a \leq n - 1$, $a^{n-1} \equiv 1 \pmod{n}$, then $n$ must be prime.

20. Observe that $1 + \frac{1}{2} = \frac{3}{2}$; $1 + \frac{1}{2} + \frac{1}{3} = \frac{7}{6}$; $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} = \frac{12}{5}$; $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{8}{3}$; $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} = \frac{15}{7}$, and so on. Conjecture a theorem that implies all these observations, and then prove that theorem.

21. Prove that: If $f(n) = \prod_{d \mid n} g(d)$ then $g(n) = \prod_{d \mid n} (f(d))^\mu(d)$.\[\sigma(n) \geq \sqrt{n}. \quad \text{[Hint: First show that } \sigma(n) \geq \prod_{d \mid n} d^{\frac{\mu(d)}{2}} \text{.]}\]

22. If $n - 1$ and $n + 1$ are twin primes with $n > 4$ then show that $\phi(n) \leq \frac{n}{3}$.

23. For a fixed positive integer $k$, if $\phi(x) = k$ has a unique integer solution, say $x = n_0$, then show that $36 \mid n_0$. [Comment: compare this to Exercise 17b in Section 7.2].

24. Observe that: $1 + 2 = \frac{3 \cdot 2}{2}$; $1 + 3 = \frac{4 \cdot 2}{2}$; $1 + 2 + 3 + 4 = \frac{5 \cdot 4}{2}$; $1 + 5 = \frac{6 \cdot 2}{2}$; $1 + 2 + 3 + 4 + 5 + 6 = \frac{7 \cdot 6}{2}$; $1 + 3 + 5 + 7 = \frac{8 \cdot 4}{2}$, and so on. Conjecture a theorem that implies all these observations, and then prove that theorem.

25. (a) First prove 8.1.6c from the textbook and then use it to prove that: there are infinitely many primes of the form $6k + 1$.
   (b) First prove 8.1.8a from the textbook and then use it to prove 8.1.9 from the textbook.
27. Prove that the product of all the primitive roots of a prime $p > 3$ is congruent to 1 modulo $p$.

28. Let $p$ be an odd prime with $\gcd(a, p) = 1$. Let $d = \gcd(m, p - 1)$. Prove that:
$x^m \equiv a \pmod{p}$ has a solution if and only if $a^{(p-1)/d} \equiv 1 \pmod{p}$. [Comment: This is a generalization of Euler’s Criterion for $m$th-power residues of an odd prime]