Graph Fall-Colouring: Some New Perspectives

Christodoulos Mitillos

Illinois Institute of Technology

cmitillo@hawk.iit.edu
Joint work with Hemanshu Kaul
Basic Definitions

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- **Dominating Set**: A set of vertices $S$, such that every vertex is in $S$ or has a neighbour in $S$. 
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Eight Queens Problem: Can we place eight queens on a regular chessboard, so that no two queens threaten each other, and every empty square is threatened by at least one queen?

Represent the chessboard as a graph; each square will correspond to a vertex, and two vertices will be adjacent if their corresponding squares are on the same vertical, horizontal, or diagonal line.

The problem is equivalent to asking whether we can find 8 vertices in this graph, forming a set which is both independent and dominating. (The answer is ”Yes”.)
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- Finding a single independent dominating set in a graph is easy and can be done with a greedy algorithm.

- Finding the smallest and largest independent dominating set in a graph are both NP-complete problems.

- Berge (1962) showed that any two independent dominating sets in a graph will not be comparable; independent dominating sets are maximal independent (and the converse holds) and minimal dominating (but the converse does not hold).
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Deciding if a given graph with maximum degree at least 4 has two disjoint independent dominating sets is NP-complete (Henning et al., 2009).

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- Introduced in 2000, by Dunbar, Hedetniemi, Hedetniemi, Jacobs, Knisely, Laskar and Rall.

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- Every colour class is a maximal independent set.
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Figure: 2- and 3-fall-colouring of $C_6$
Graph Fall-Colouring

Figure: $C_5$ cannot be fall-coloured
**Definition**

The *Fall Set* of a graph $G$, $\text{Fall}(G)$ is the set of all $k$, such that $G$ has a fall-colouring with $k$ colour classes. $G$ is said to be *$k$-fall-colourable* if $k \in \text{Fall}(G)$.
The following families of graphs have the special property that $\text{Fall}(G) = \{\chi(G)\}$, iff $\chi(G) = \delta(G) + 1$:

- Threshold Graphs (M., Kaul)
- Split Graphs (M., Kaul)
- Strongly Chordal Graphs (Lyle et al. [2005])
- Complete Graphs (Dunbar et al. [2000])
- Bipartite Graphs with at least one leaf (Dunbar et al. [2000])
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All the graphs in the preceding slide are chordal graphs, which are perfect graphs.

**Conjecture 1:**
If $G$ is a perfect graph with $\chi(G) = \delta(G) + 1$, then $Fall(G) = \{\chi(G)\}$.

**Conjecture 2:**
If $G$ is a chordal graph with $\chi(G) = \delta(G) + 1$, then $Fall(G) = \{\chi(G)\}$. 
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Theorem [M., Kaul]

Let \( G \) be \( k \)-fall-colourable and \( H \) be \( k \)-colourable. Then \( G \square H \) is \( k \)-fall-colourable.

Let \( g : V(G) \mapsto [k] = \{0, 1, \ldots, k - 1\} \) be a \( k \)-fall-colouring and \( h : V(H) \mapsto [k] \) be a \( k \)-colouring. Then \( f : V(G) \times V(H) \mapsto [k] \), where \( f(u, v) = (g(u) + h(v)) \mod k \) is a \( k \)-fall-colouring.
Products of Graphs

Theorem [M., Kaul]

Let $G$ be $k$-fall-colourable and $H$ have no isolated vertices. Then $G \times H$ is $k$-fall-colourable.

Let $g$ be a $k$-fall-colouring of $G$. Then $f(u, v) = g(u)$ is a $k$-fall-colouring of $G \times H$. 
Let $S = \{s_1, s_2, \ldots, s_r\}$ be a set, so that $s_i \neq 1, \forall i$.

**Theorem [Dunbar et al., 2000]**

Let $G = K_{s_1} \times K_{s_2} \times \ldots \times K_{s_r}$. Then $S \subseteq \text{Fall}(G)$.

Each cardinal $(r - 1)$-dimensional hyperplane is an independent dominating set.

Colouring along these hyperplanes yields a fall-colouring. Valencia-Pabon (2010) and Klavžar and Melkiš (2011) fully characterised the independent dominating sets of all such graphs, for $r \leq 4$, thus showing under what conditions $S \neq \text{Fall}(G)$. 
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Let $S = \{s_1, s_2, \ldots, s_r\}$ be a multiset, so that $s_i \neq 1, \forall i$.

**Theorem [M., Kaul]**

Let $G = K_{s_1} \square K_{s_2} \square \ldots \square K_{s_r}$. A subset of $V(G)$ is an independent dominating set iff it corresponds to $s_i$ vertices which share the same coordinates, except on the $i^{th}$ position.

The vertices of the graph form an $r$-dimensional integer lattice. Edges connect vertices which are not in the same cardinal line. This gives a complete characterisation of all independent dominating sets in our graph.
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Let $G = K_{s_1} \boxtimes K_{s_2} \boxtimes \ldots \boxtimes K_{s_r}$. A subset of $V(G)$ is an independent dominating set iff it corresponds to $s_i$ vertices which share the same coordinates, except on the $i^{th}$ position.

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Constructions for Graphs with specified Fall Sets

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Theorem [M., Kaul]

Let $|S| = 2$. Then \( \text{Fall}(G) = S \).

Theorem [M., Kaul]

Let $|S| = 3$. Then \( k \in \text{Fall}(G) \) iff \( k \) can be expressed as the sum of \( s_i \) summands, each taking a value in \( S \setminus \{s_i\} \), for some \( i \).

For example, when \( S = \{2, 3, 4\} \), \( \text{Fall}(G) = \{6, 7, \ldots, 12\} \). On the other hand, when \( S = \{2, 3, 5\} \), \( \text{Fall}(G) = \{6, 8, 9, \ldots, 15\} \).
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Corollary [M., Kaul]

Let $a > 1$, $k \geq 1$, and define $b = a + k$ and $c = a + 2k$. Then

$$\text{Fall}(K_a \square K_b \square K_c) = \{a^2 + rk | a \leq r \leq 3a + 2k\}.$$ 

For any $k$, and any quadratic residue $x$ of $k$, we can construct a graph whose Fall set consists entirely of integers congruent to $x$, modulo $k$.

This also means that we can create graphs with discontiguous Fall sets, making the gaps as large as we like.
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Example: \( \text{Fall}(K_2 \square K_4 \square K_6) = \{4 + 2r | 2 \leq r \leq 10\} = \{2t | 4 \leq t \leq 12\}. \)

Example: \( \text{Fall}(K_3 \square K_5 \square K_7) = \{9 + 2r | 3 \leq r \leq 13\} = \{2t + 1 | 7 \leq t \leq 17\}. \)

Example: \( \text{Fall}(K_1 3 \square K_1 8 \square K_2 3) = \{169 + 5r | 13 \leq r \leq 49\} = \{5t - 1 | 47 \leq t \leq 83\}. \)
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Example: \( \text{Fall}(K_1 \Box K_1 \Box K_2) = \{ 169 + 5r | 13 \leq r \leq 49 \} = \{ 5t - 1 | 47 \leq t \leq 83 \} \).
Sets with Different Chromatic and Fall-Chromatic Numbers

Question 1 (Dunbar et al. [2000]): Can \( \min(Fall(G)) - \chi(G) \) be made arbitrarily large?

**Theorem [M., Kaul]**

Let \( k \geq 3 \) and \( t > k \). Then there exists a graph \( G \) with \( \chi(G) = k \) and \( Fall(G) = \{t\} \).

We modify the graph \( K_k \times K_t \), by removing the edges of ones \( t \)-star.
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Theorem [Bollobás, 1978]

Let $G$ be a $k$-colourable graph on $n$ vertices, with $\delta(G) > \frac{k-2}{k-1} n$. Then $\chi(G) = k$.

Theorem [Bollobás, 1978]

Let $G$ be a $k$-colourable graph on $n$ vertices, with $\delta(G) > \frac{3k-5}{3k-2} n$. Then $G$ is uniquely $k$-colourable.
Uniquely Colourable Graphs

Proposition

Let $G$ be uniquely $k$-colourable. Then $k \in \text{Fall}(G)$.

Since there is a unique $k$-colouring, every vertex has a neighbour in every colour class, other than its own.
Theorem [M., Kaul]

Let $G$ be a $k$-colourable graph on $n$ vertices, with $\delta(G) > \frac{k-2}{k-1}n$. Then every $k$-colouring of $G$ is a fall-colouring.

Assume that some vertex is not dominated by some colour class and use the pigeonhole principle, to arrive at a contradiction.
Theorem [M., Kaul]

Let \( G \) be a \( k \)-colourable, regular graph on \( n = r(k - 1) \) vertices, with every vertex having degree \( \frac{k-2}{k-1} \) \( n = r(k - 2) \). Then either:

1) Every \( k \)-colouring of \( G \) is a \( k \)-fall-colouring, or
2) Any \( k \)-colouring of \( G \) which is not a \( k \)-fall-colouring can be converted into a \((k - 1)\)-fall-colouring, by merging two colour classes.

Moreover, for any valid \( k \) and \( n \), there exists such a graph with no \( k \)-fall-colouring.

The graph which shows the sharpness of the previous result is the Turán Graph, \( T(n, (k - 1)) \).
Graphs with Fall- and Non-Fall-Colourings

Theorem [M., Kaul]

For all $k \geq 3$, there exists a graph $G_k$, such that $\chi(G_k) = k$, $k \in \text{Fall}(G_k)$, and $G_k$ has a $k$-colouring, which is not a $k$-fall-colouring.

Create $k - 1$ $k$-cliques. Then add a new vertex, and connect it to one vertex from each clique.
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Theorem [M., Kaul]

Let $r \geq 2$, $k \geq 3$, and $k \geq r$. Then, there exists a graph $G_{k,r}$, such that $\chi(G_{k,r}) = r$, $k \in \text{Fall}(G_{k,r})$, and $G_{k,r}$ has a $k$-colouring, which is not a $k$-fall-colouring.

Take the Categorical Product of $G_k$ from the previous result and $K_r$. 
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Applied Scenarios

- City planning: Where to place emergency units, for maximum coverage and minimal cost.

- Radio signals: How to allocate channels to transceivers, for maximum coverage and minimal interference.
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A proper colouring is a $(0, 0, \ldots, 0)$-near-colouring.

We can create a similar notion, for fall-colouring.
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We say that a graph $G$ is $(k_1, k_2, \ldots, k_r)$-fall-near-colourable, if there exists a partition of $V(G)$ into $r$ sets, such that the $i^{th}$ set induces a subgraph with maximum degree no more than $k_i$, and is dominating.

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- $(2)$-fall-near-colourable, for any $n$.
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- $(0, 0)$-fall-near-colourable, if $2 \mid n$.
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For example, $K_4$ is $(0, 0, 0, 0)$-, $(1, 1, 1)$-, $(1, 1)$-, $(2, 2)$- and $(3)$-fall-near-colourable.

These are not always optimal; in the above example, we also have that $K_4$ is $(1, 0, 0)$-, and $(2, 0)$-fall-near-colourable.
Fall-Near-Colouring

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Theorem [M., Kaul]

If $G$ is $(a_1, a_2, \ldots, a_k)$-fall-near-colourable and $H$ is $(b_1, b_2, \ldots, b_k)$-near-colourable, then $G \Box H$ is $(m_1, m_2, \ldots, m_k)$-fall-near-colourable, where

$$m_r = \max_{i+j \equiv r \pmod{k}} (a_i + b_j).$$

If $f$ and $g$ are given colourings of $G$ and $H$ respectively, then give $(u, v)$ the colour $f(u) + g(v) \mod k$.

If $u$ has $x$ incident monochromatic edges, and $v$ has $y$ incident monochromatic edges, $(u, v)$ will have exactly $x + y$ incident monochromatic edges.
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$(u, v)$ will have the same colour as $u$. Furthermore, every monochromatic edge incident to $u$ generates $\Delta(H)$ monochromatic edges incident to $(u, v)$. 
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Conjecture: If $G$ is a perfect graph (chordal graph) with $\chi(G) = \delta(G) + 1$, $\text{Fall}(G) = \{\chi(G)\}$.

What is the tradeoff between colour class dependence and minimum degree?

What other fall-near-colourings are there for complete graphs? For products of graphs? For other families of graphs?

Can we convert a fall-colouring of a graph to a fall-near-colouring? Is there a meaningful bound to the maximum dependence this will produce?

Can we get a fall-near-colouring of a graph from a near-colouring?

What conditions are enough to ensure that an arbitrary graph has a fall-near-colouring, with a given number of colour classes?
Open Questions/Future Work

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Thank you.
Questions? Comments?