Counting List Colorings: Two new perspectives on a theme of Kostochka and Sidorenko

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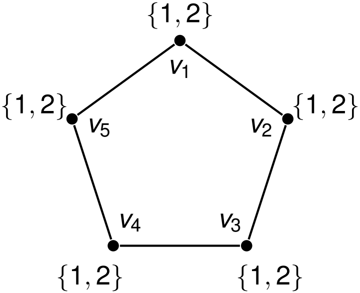
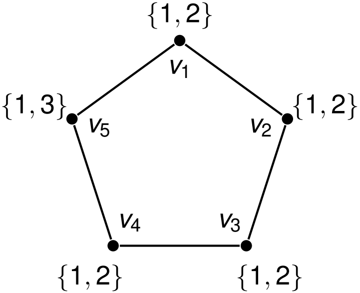
based on joint works with
Jeffrey Mudrock (U. South Alabama)

Classic Coloring vs List Coloring





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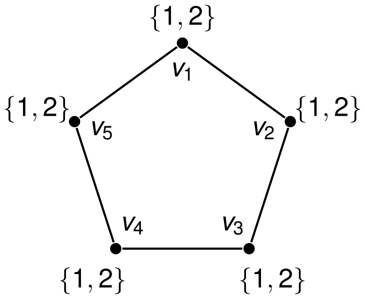

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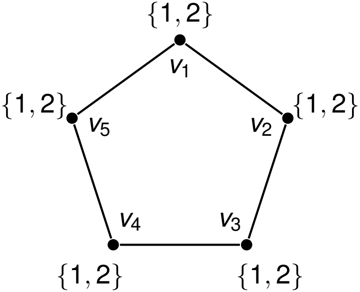

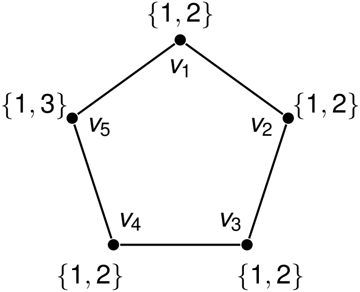

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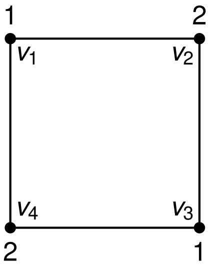
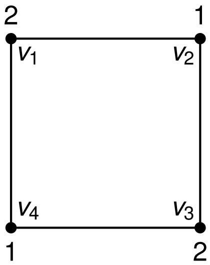
Not true: $\chi(K_{a,a^a}) = 2$ but $\chi_\ell(K_{a,a^a}) = a + 1$.

Chromatic Polynomial





Birkhoff 1912: For $q \in \mathbb{N}$, $P(G, q)$ denotes the number of proper colorings of G with colors from $\{1, ..., q\}$.

$P(C_4, 2) = 2$

$P(C_n, q) = (q-1)^n + (-1)^n(q-1)$.

List Color Function

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- Kostochka, Sidorenko (1990): $P_\ell(G, q) = P(G, q)$ for all q, if G is chordal.
  Kirov, Naimi (2016): $P_\ell(C_n, q) = P(C_n, q)$ for all q.
  Notion of Enumeratively Chromatic Choosable Graphs proposed by Kaul, Mudrock, et al. (2023), formally studied by Allred and Mudrock (2025+).

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- In general, $P_\ell(G, q) \leq P(G, q)$.
- $P(K_{2,4}, 2) = 2$, and yet $P_\ell(K_{2,4}, 2) = 0$.
- $P_\ell(K_{3,26}, 3) \leq 3^8 2^{12} < 3^1 2^{26} \leq P(K_{3,26}, 3)$.

Kostochka and Sidorenko's Question

Kostochka, Sidorenko (1990): Given any graph G, does there exist $\tau(G) \in \mathbb{N}$ such that $P_\ell(G,q) = P(G,q)$ for all $q > \tau(G)$.

- Donner (JGT 1992): $\tau(G) < \infty$
- Thomassen (JCTB 2009): $\tau(G) < |V(G)|^{10} + 1$
- Wang, Qian, Yan (JCTB 2017): $\tau(G) < 1.135(|E(G)| - 1)$
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Theorem (Dong, Zhang (2023))

Let G be a simple graph with n vertices and $m \geq 4$ edges. Then, for any q-assignment L of G with $q \geq m - 1$,
$P(G, L) - P(G, q) \geq$
$((q - m + 1)q^{n-3} + (q - m + 3)(c/3)q^{n-5}) \sum_{uv \in E(G)} |L(u) - L(v)|$,
where $c \geq (q - 1)(q - 3)/8$.

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Conjecture (Dong, Zhang (2023))

$\tau(G) < O(|V(G)|)$

$\tau(G) < O(|\Delta(G)|)$

Theme of Kostochka and Sidorenko

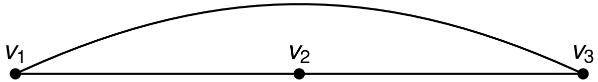
An enumerative function of (a variant of) list colorings equals the corresponding enumerative function of (the same variant of) classical colorings, when the number of colors is large enough.

This has also been explored for DP (correspondence) colorings of graphs, and colorings of signed graphs.

In this talk -

- Packings of list colorings
- List coloring of unlabeled graphs

Packing of List Colorings

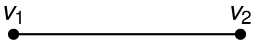


$\{1,2,3\}$	$\{1,3,4\}$	$\{2,3,4\}$
1	3	4
2	1	3
3	4	2

Cambie, C v Batenburg, Davies, Kang (RSA 2024): For graph G with list assignment L, L-packing of size k of G is a set of k L-colorings of G, $\{f_1, \ldots, f_k\}$, such that $f_i(v) \neq f_j(v)$ whenever $i, j \in [k]$, $i \neq j$, and $v \in V(G)$.

The list packing number of G, $\chi^{\star}_{\ell}(G)$, is the least q such that G has a L-packing of size q for every q-assignment L.

Counting Packings of List Colorings

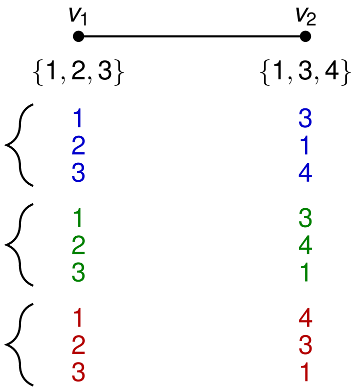


$\{1, 2, 3\}$	$\{1, 3, 4\}$
1	3
2	1
3	4
1	3
2	4
3	1
1	4
2	3
3	1

$P^{\star}(G, L, k)$ denotes the number of L-packings of size k of G.

$P^{\star}(K_2, L, 3) = 3$ for the 3-assignment L above.

Counting Packings of List Colorings



$P^{\star}(G, L, k)$ denotes the number of L-packings of size k of G.

For $k \leq q$, $P^{\star}_{\ell}(G, q, k)$, the (q, k)-fold list packing function of G, is the minimum value of $P^{\star}(G, L, k)$ over all q-assignments L.

List Packing Function

Recall, for graph G with list assignment L, an L-packing of size k is a set of k pairwise disjoint L-colorings of G. $P^\star(G, L, k)$ is the number of such L-packings of size k.

For $k \leq q$, $P_\ell^\star(G, q, k)$ is the minimum value of $P^\star(G, L, k)$ over all q-assignments L.

- The list color function $P_\ell(G, q)$ equals $P_\ell^\star(G, q, 1)$.

Classical Packing Function

For $k \leq q$, $P^\star(G, q, k)$, the (q, k)-fold classical packing function of G, is $P^\star(G, L, k)$ for L that assigns the list $[q]$ to each vertex of G.

- The chromatic polynomial $P(G, q)$ equals $P^\star(G, q, 1)$.

v_1	v_2
$\{1, 2, 3\}$	$\{1, 2, 3\}$
1	2
2	3
3	1
1	3
2	1
3	2

$P^\star(K_2, 3, 3) = 2$

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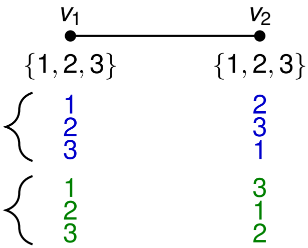


$$P^\star(K_2, 3, 3) = 2$$

Thematic Question for List Packings

- For $k \leq q$, $P^\star(G,q,k)$ is $P^\star(G,L,k)$ for L that assigns the list $[q]$ to each vertex of G.

  The chromatic polynomial $P(G,q)$ equals $P^\star(G,q,1)$.

- For $k \leq q$, $P^\star_\ell(G,q,k)$ is the minimum value of $P^\star(G,L,k)$ over all q-assignments L.

  The list color function $P_\ell(G,q)$ equals $P^\star_\ell(G,q,1)$.

- $P^\star_\ell(G,q,k) \leq P^\star(G,q,k)$ for all $k \leq q$.

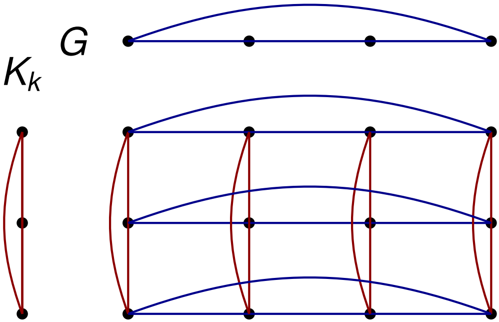
Question (K., Mudrock (2025))

For every graph G does there exist an $N \in \mathbb{N}$ such that $P^\star_\ell(G,q,k) = P^\star(G,q,k)$ whenever $k \leq q$ and $q \geq N$? The case $k = q$ is of particular interest.

A Key Idea

Observation (based on Mudrock (2023))

Suppose L is a q-assignment for graph G and $L^{(k)}$ is the list assignment for $G \square K_k$ given by $L^{(k)}(v, w_i) = L(v)$ for each $v \in V(G)$ and $i \in [k]$. G has an L-packing of size k if and only if there is an $L^{(k)}$-coloring of $G \square K_k$.



Results for Counting List Packings

For every graph G does there exist an $N \in \mathbb{N}$ such that $P_\ell^\star(G, q, k) = P^\star(G, q, k)$ whenever $k \leq q$ and $q \geq N$? The case $k = q$ is of particular interest.

Theorem (K., Mudrock (2025))

If T is a tree on n vertices and $q \in \mathbb{N}$, then $P_\ell^\star(T, q, q) = P^\star(T, q, q) = (!q)^{n-1}$.

$!q$ denotes the number of derangements of $[q]$.

Results for Counting List Packings

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Lemma (K., Mudrock (2025))

For any graph G, and $q, k \in \mathbb{N}$ satisfying $k \leq q$ $P^\star(G, q, k) = \frac{P(G \square K_k, q)}{k!}$.

Corollary (K., Mudrock (2025))

$P^\star(K_n, q, k) = 0$ whenever $k \leq q \leq n$, and $P^\star(K_n, q, k) = \frac{L(n,k,q)}{k!}$ whenever $q \geq n$ and $k \leq q$ where $L(n, k, q)$ denotes the number of $n \times k$ Latin arrays containing at most q symbols.

Results for Counting List Packings

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Theorem (K., Mudrock (2025))

Suppose G is an n-vertex graph with m edges. If $q, k \in \mathbb{N}$ satisfy $q \geq nk(k-1)/2 + mk - 1$, then $P_\ell^\star(G, q, k) = P^\star(G, q, k)$.

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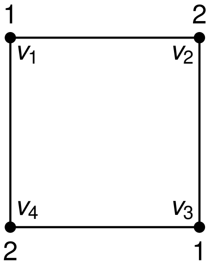
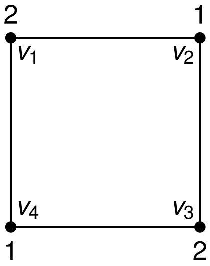
- Note that plugging in $k = 1$ recovers the Dong-Zhang (2023) result: $P_\ell(G, q) = P(G, q)$ for $q > m - 1$.
- Note that our result requires q to be at least quadratic in k. As a next step, improve this bound on q to a linear ftn of k.

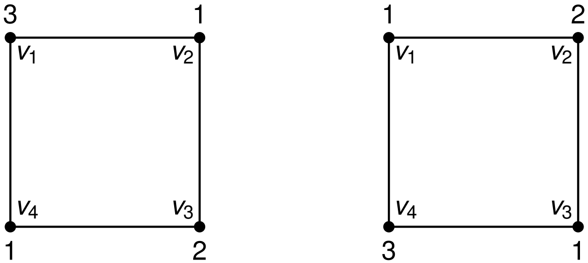
Coloring under Symmetries





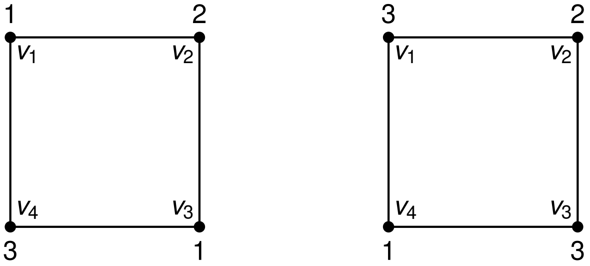
- The above two 2-colorings of C_4 are the same under the automorphism of C_4 given by 90°-rotation (in permutation form $\pi = (v_1\ v_2\ v_3\ v_4)$). $f(\pi(v)) = g(v)\ \forall v$.

Coloring under Symmetries



- The above two 3-colorings of C_4 are the same under the automorphism of C_4 given by 90°-rotation (in permutation form $\pi = (v_1\ v_2\ v_3\ v_4)$). $f(\pi(v)) = g(v)\ \forall v$.

Coloring under Symmetries



- The above two 3-colorings of C_4 are distinct.
- Note that the names of the colors do not change. They are not interchangeable.

Coloring of Unlabeled Graphs

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- Symmetric group S_n acts on the set G_n of labeled n-vertex graphs. $\mathcal{G}$ is an orbit of this action.

Coloring of Unlabeled Graphs

- Unlabeled graph $\mathcal{G}$ is an "isomorphism class of labeled graphs". Its a set of labeled graphs.
- Suppose $G \in \mathcal{G}$. Let $U(G, q)$ be the set of proper q-colorings of G.
- For $f, g \in U(G, q)$, $f \sim g$ if there exists $\pi \in \mathrm{Aut}(G)$ s.t. $f\pi = g$, that is $f(\pi(v)) = g(v) \;\; \forall v \in V(G)$.

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- Let $\mathcal{G}_{C_4}$ be the unlabeled C_4. Then $P(\mathcal{G}_{C_4}, 2) = 1$.

Orbit Counting

- For $G \in \mathcal{G}$ and $\pi \in \mathrm{Aut}(G)$, (π, q)-coloring is a proper q-coloring, f, of G with the property that $f(\pi(v)) = f(v)$ for each $v \in V(G)$.
- Let $P(G, \pi, q)$ be the number of proper (π, q)-colorings of G.

Theorem (Hanlon (1985))

For any $q \in \mathbb{N}$ and $G \in \mathcal{G}$,
$P(\mathcal{G}, q) = \frac{1}{|\mathrm{Aut}(G)|} \sum_{\pi \in \mathrm{Aut}(G)} P(G, \pi, q)$.

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- Apply Burnside's lemma/ Orbit counting lemma under the set-up "Symmetric group S_n acts on the set G_n of labeled n-vertex graphs".

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- $P(\mathcal{G}, q)$ is polynomial in q of degree n with leading coefficient $1/|\mathrm{Aut}(G)|$.
- Computing $P(G, \pi, q)$ for each $\pi \in \mathrm{Aut}(G)$ (expressed as permutation in cyclic form) gives a systematic method for computing $P(\mathcal{G}, q)$.

List coloring unlabeled graphs

- Suppose $G \in \mathcal{G}$, and L be a list assignment for G. Let $U(G, L)$ be the set of proper L-colorings of G.

- For $f, g \in U(G, L)$, $f \sim g$ if there exists $\pi \in \mathrm{Aut}(G)$ s.t. $f\pi = g$, that is $f(\pi(v)) = g(v) \ \ \forall v \in V(G)$.

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- $u_\ell(G, L)$ be the number of equivalence classes under the relation $\sim$ on $U(G, L)$.
- $P_\ell(\mathcal{G}, q)$, the unlabeled list color function, be the minimum value of $u_\ell(G, L)$ over all q-assignments L of an arbitrarily chosen $G \in \mathcal{G}$.

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- $P_\ell(\mathcal{G}, q) \leq P(\mathcal{G}, q)$. Since $u_\ell(G, C) = P(\mathcal{G}, q)$, where C assigns the list $[q]$ to all vertices.

Thematic Question for Unlabeled List Color Ftn

- $P_\ell(\mathcal{G}, q)$ is the unlabeled list color function.
- $P(\mathcal{G}, q)$ is the unlabeled chromatic polynomial.

- $P_\ell(\mathcal{G}, q) \leq P(\mathcal{G}, q)$.

Question (K., Mudrock (2024+))

For which unlabeled graphs $\mathcal{G}$ does there exist an $N \in \mathbb{N}$ so that $P_\ell(\mathcal{G}, q) = P(\mathcal{G}, q)$ whenever $q \geq N$?

Coloring Point-Determining Graphs

Theorem (K., Mudrock (2024+))

If $\mathcal{G}$ is an unlabeled, connected, point-determining graph, then there exists an $N \in \mathbb{N}$ such that $P_\ell(\mathcal{G}, q) = P(\mathcal{G}, q)$ whenever $q \geq N$.

- A graph G is called point-determining if no two distinct vertices in G have the same neighborhood in G.
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Coloring Point-Determining Graphs

Theorem (K., Mudrock (2024+))

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  e.g. Complete graphs, Asymmetric graphs.
- Point-determining graphs (aka Irreducible graphs and Mating graphs) have long been studied in the context of various graph colorings, graph homomorphisms, graph domination, and mating systems since at least 1961.
- Unlabeled point-determining (connected) graphs have been enumerated explicitly using different methods.
  Almost all graphs are asymmetric and hence point-determining.

Extensions of ideas of Hanlon

- Given $G \in \mathcal{G}$, $\pi \in \mathrm{Aut}(G)$, L a q-assignment of G.
  A proper (π, L)-coloring is a proper L-coloring, f, of G such that $f(\pi(v)) = f(v)$ for each $v \in V(G)$.
  $P(G, \pi, L)$ is the number of proper (π, L)-colorings of G.

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Lemma (K., Mudrock (2024+))

Suppose L is an k-assignment for $G \in \mathcal{G}$. Then, $u_\ell(G, L) \geq \frac{1}{|\mathrm{Aut}(G)|} \sum_{\pi \in \mathrm{Aut}(G)} P(G, \pi, L)$.

Extensions of ideas of Hanlon

- For $\pi \in \mathrm{Aut}(G)$ and $C_1 \ldots C_s$ the cycle decomposition of π. The quotient of G with respect to π, denoted $G : \pi$, is the graph with vertex set $\{C_1, \ldots, C_s\}$ and edges so that $C_iC_j \in E(G : \pi)$ if and only if there is a $u \in C_i$ and $v \in C_j$ such that $uv \in E(G)$.

Extensions of ideas of Hanlon

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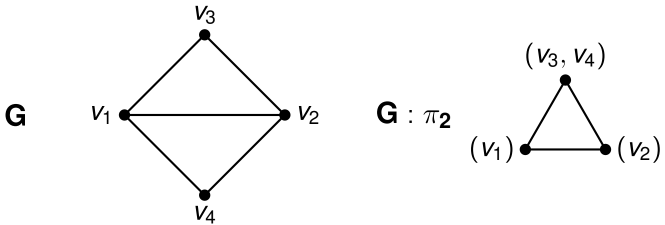

$\mathrm{Aut}(G) = \{\pi_1, \pi_2, \pi_3, \pi_4\}$, where $\pi_1 = (v_1\ v_2)(v_3)(v_4)$,
$\pi_2 = (v_1)(v_2)(v_3\ v_4)$, $\pi_3 = (v_1\ v_2)(v_3\ v_4)$, $\pi_4 = (v_1)(v_2)(v_3)(v_4)$.

$P(G, \pi_1, q) = 0$; v_1, v_2 adjacent but need same color by π_1.
$P(G, \pi_2, q) = P(G : \pi_2, q) = P(K_3, q) = q(q-1)(q-2)$.

Extensions of ideas of Hanlon

Lemma (K., Mudrock (2024+))

If L is a q-assignment for G, then
(i) If there is an $i \in [s]$ such that C_i contains two adjacent vertices in G, then $P(G, \pi, L) = 0$.
(ii) Otherwise, $P(G, \pi, L) = P(G : \pi, L')$ where L' is the list assignment for $G : \pi$ given by $L'(C_i) = \bigcap_{v \in C_i} L(v)$ for each $i \in [s]$.

Main Theorem

Theorem (K., Mudrock (2024+))

Suppose $\mathcal{G}$ is an unlabeled, connected graph of order n and size $m \geq 4$ with $G \in \mathcal{G}$. Suppose for each $\pi \in \mathrm{Aut}(G) - \{id\}$ with cycle decomposition $C_1 \ldots C_s$ we have: If for each $i \in [s]$ the vertices in C_i are pairwise nonadjacent in G, then $s \leq n - 2$. Then, there exists an $N \in \mathbb{N}$ such that $P_\ell(\mathcal{G}, q) = P(\mathcal{G}, q)$ whenever $q \geq N$.

Since G is point-determining if and only if all transpositions in $\mathrm{Aut}(G)$ interchange two adjacent vertices.

Corollary

If $\mathcal{G}$ is an unlabeled, connected, point-determining graph, then there exists an $N \in \mathbb{N}$ such that $P_\ell(\mathcal{G}, q) = P(\mathcal{G}, q)$ whenever $q \geq N$.

Non-point-determining Graphs

Theorem (K., Mudrock (2024+))

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The operation of taking the join of an appropriate point-determining graph (e.g., complete graphs and asymmetric graphs) with two nonadjacent vertices will preserve this property. Note the new graph is not point-determining.

Non-point-determining Graphs

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Theorem (K., Mudrock (2024+))

Suppose $\mathcal{G}$ is an unlabeled graph of order n with n possibly zero. Suppose $G \in \mathcal{G}$. Suppose that G is point-determining and for each $\pi \in \mathrm{Aut}(G) - \{id\}$ if $C_1 \dots C_s$ is the cycle decomposition of π, then there is an $i \in [s]$ such that C_i contains at least two adjacent vertices.
Let $G' = G \vee \overline{K_2}$. Also, let $\mathcal{G}'$ be the unlabeled graph such that $G' \in \mathcal{G}'$. Then, there exists an $N \in \mathbb{N}$ such that $P_\ell(\mathcal{G}', q) = P(\mathcal{G}', q)$ whenever $q \geq N$.

What about disconnected graphs?

Proposition (K., Mudrock (2024+))

Suppose G_1 and G_2 are connected graphs that are non-isomorphic and vertex disjoint. Let G be the disjoint union of G_1 and G_2. Suppose $\mathcal{G}$, $\mathcal{G}_1$, and $\mathcal{G}_2$ are the unlabeled graphs corresponding to G, G_1 and G_2, respectively. Then, $P_\ell(\mathcal{G}, q) = P_\ell(\mathcal{G}_1, q)\, P_\ell(\mathcal{G}_2, q)$.

- What about unlabeled disconnected graphs that have two distinct, isomorphic, connected components?

What about disconnected graphs?

What about unlabeled disconnected graphs that have two distinct, isomorphic, connected components? Say, each component is a single vertex?

- Using Hanlon's Theorem: if $\overline{K}_n$ is an unlabeled, edgeless n-vertex graph with $n \geq 2$, then for each $q \in \mathbb{N}$, $P(\overline{K}_n, q) = \frac{1}{n!} \prod_{i=0}^{n-1} (q+i) \neq q^n = P(\overline{K}_1, q)^n$.
- This observation along with our work implies that there is a $q \in \mathbb{N}$ such that $P_\ell(\overline{K}_2, q) \neq P_\ell(\overline{K}_1, q)^2$.

A New “Shameful Conjecture”

- What about unlabeled disconnected graphs that have two distinct, isomorphic, connected components? Say, each component is a single vertex?

Conjecture (K., Mudrock (2024+))

Let $\overline{K}_n$ be an unlabeled, edgeless n-vertex graph. For each $n \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that $P_\ell(\overline{K}_n, q) = P(\overline{K}_n, q)$ whenever $q \geq N$.

- Known only for $n = 1, 2$.

Thank You!

Questions?

- For every graph G, does there exist an $N \in \mathbb{N}$ such that $P_\ell^\star(G, q, k) = P^\star(G, q, k)$ whenever $k \leq q$ and $q \geq N$?
  What about $q = k$?
  What about $q \geq o(k^2)$?

- For every unlabeled graph $\mathcal{G}$, does there exist an $N \in \mathbb{N}$ so that $P_\ell(\mathcal{G}, q) = P(\mathcal{G}, q)$ whenever $q \geq N$?

- Conjecture [K., Mudrock (2024+)]
  Let $\overline{K}_n$ be an unlabeled, edgeless n-vertex graph. For each $n \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that $P_\ell(\overline{K}_n, q) = P(\overline{K}_n, q)$ whenever $q \geq N$.

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